Resource Investment, Allocation and Product Pricing in Mass Customization with Secondary Market

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Abstract

We consider an n-product firm that invests in m resources under demand uncertainty, and sets prices, produces to order and trades excess resources in the secondary market after demand uncertainty is resolved. After characterizing the firm's optimal strategy and expected profit for a very general input-output transformation technology, we focus on a mass-customized product, such as a car or a PC, that is assembled from a number of components all of which exist in several versions. We derive a series of new and highly counterintuitive insights into the value drivers and optimal pricing in mass customization. For example, we show that the firm's expected profit increases in demand correlation even if the corresponding products share common components, as long as the degree of commonality does not exceed a threshold. We also show that a high degree of commonality between two products is a reason for their optimal prices to move in the same direction. Finally, the novel modelling construct which enables us to obtain all of these results analytically contributes to the operations research methodology.

1 Introduction

The last decade has witnessed an unprecedented proliferation of mass customization in many industries ranging from car and computer manufacturing to services such as banking. The idea of mass customization is deceptively simple: efficiently make to order what customers want. In practice, being efficient at managing skyrocketing product or service variety is not an easy task and it poses considerable challenges to resource management. Consider a product such as a personal computer

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that is built to order from 10 parts each of which is available in 3 versions: the corresponding bill of material involves 30 parts that can be combined into 59,049 product configurations. John Deere manufactures a seed planter that can be custom ordered in over 6 million possible configurations. Such high product variety, combined with competitive pressures on cutting inventory cost, delivery lead-times and product life-cycles, makes matching supply and demand very challenging and, at the same time, critical to a firm's success.

At the core of mass customization is the postponement of the final product differentiation or assembly until customer demands are known. This postponement reduces demand uncertainty by shifting it from output to input levels, taking advantage of statistical aggregation of demands for common inputs. Thus, instead of forecasting demand for each of 59,049 different product configurations, the computer manufacturer has to forecast demand for only 30 components. While production (i.e., assembly or differentiation) postponement makes it easier for supply to meet demand, short-term pricing flexibility enables the firm to influence demand to meet supply. A distribution channel that particularly facilitates such responsive pricing is e-commerce, but the overall use of information technology has enabled short-term price flexibility in other channels as well.

While production postponement and pricing flexibility can mitigate the mismatch between supply and demand considerably, they cannot eliminate it entirely. As a result of firms' inability to accurately forecast demand, highly liquid internet-based spot markets for excess inventory and capacity have emerged in many industries, particularly those where the product life-cycles are short. In the computer and electronics industry, for example, on-line exchanges such as Converge, America II and Smith & Associates provide access to billions of dollars worth of components. By offering continuous availability and short lead-times, such spot markets help manufacturers absorb demand shocks as well as take advantage of price arbitrage. Furthermore, according to a recent survey by Grey et al. (2005), "trading on exchanges is not limited to traditional physical commodities. It can also involve products and services such as flexible production capacity, labor, transportation and advertising."

This paper develops a conceptual model of a mass customization strategy that (i) captures the complexity of production requirements resulting from the extreme product variety, (ii) reflects pricing flexibility, and (iii) considers the effect of a secondary market. We consider a firm that has to choose inventory or capacity levels of multiple inputs, which will be used in the production of a variety of products with short life-cycles. The demand curves for these products are assumed to be linear and exposed to additive uncertainty drawn from an arbitrary multivariate distribution.

Once the demand curve uncertainty is resolved, the firm sets product prices and produces to order. At this time, the firm has the opportunity to purchase or sell extra inputs in a secondary market. The demand (supply) curves in the secondary market are also linear and downward (upward) sloping. Therefore, as the volume of the firm's secondary market trading increases, it takes place at increasingly unfavorable prices for the firm. We characterize the firm's optimal strategy and expected profit for a very general input-output transformation technology – any of the n products may use any combination of any of the m inputs. For concreteness and tractability, we then focus on a mass-customized product, such as a PC or a car, that is assembled from k components (or modules), each of which is available in l versions.

The common intuition supported by the existing academic literature (see e.g., Swaminathan and Lee 2003) suggests that the benefits of production postponement (risk-pooling) are increasing in demand variability and decreasing in correlation between demands for products that rely on common inputs. We challenge this intuition by showing that the effects of both demand variability and correlation depend critically on the commonality structure of the entire product line. For example, we prove that the benefits of production postponement and the firm's expected profit increase in demand correlation if and only if the degree of commonality between the two products does not exceed a threshold. This has important implications for the optimal pricing policy. We show that even if demand levels for different products are stochastically independent and there are no cross-price effects, the optimal prices will be correlated to induce the desired demand correlations. In particular, we show that, contrary to intuition, the prices of products with a relatively high (low) degree of commonality should move in the same (opposite) direction. Clearly, the same logic can be applied to other marketing levers, such as advertising, that influence demand but are not captured in our model.

Our ability to analyze the problem at the given level of generality relies on a novel but rather stylized model formulation. Nevertheless, the intuitive explanations that we provide for all of our "counterintuitive" results suggest the results' validity beyond our specific modelling construct. Finally, we believe that this modelling construct can be applied to other otherwise untractable problems in operations management.

2 Relation to the literature

Price and production postponement under a stochastic demand curve were analyzed in a comprehensive framework in a single product scenario by Van Mieghem and Dada (1999). The literature on product differentiation postponement has been extensive, but has typically assumed given prices and rather simple product/process designs. The seminal work of Lee (1996) and Lee and Tang (1997) focuses on process redesign to postpone product differentiation. Lee (1996) analyzes the impact of this postponement on the reduction of inventory, while Lee and Tang (1997) capture both the cost and benefits of redesigning a process for postponement. A comprehensive review of the postponement literature can be found in Swaminathan and Lee (2003).

A general conclusion of the existing literature is that the value of postponement is increasing in demand variability and decreasing in demand correlation (e.g., Lee 1996, Lee and Tang 1997). We challenge this common perception by showing that both of these effects depend critically on the degree of commonality between various product configurations. In particular, we prove that the value of postponement is increasing in demand correlation when the two products share a relatively low number of inputs. Similarly, the effect of demand variability on the benefits of postponement depends on the commonality structure throughout the whole product line.

Closely related to product differentiation postponement is the use of component commonality and assemble-to-order systems. The classical work in this field includes Collier (1982), Baker et al. (1986), Gerchak et al. (1988), and Gerchak and Henig (1989), all of whom analyze the inventory savings due to commonality. With respect to market drivers of commonality, the current literature finds that the value of commonality is increasing in demand variability and decreasing in demand correlation; see e.g., Groenevelt and Rudi (2000). (Gerchak and He 2003 show, assuming one input and two demand classes, that the benefits of risk pooling may be decreasing in demand variability for some, rather irregular, demand distributions but are guaranteed to increase in demand variability under a structured model of increased variability.)

The newsvendor-based models on input commonality have been generalized by Van Mieghem and Rudi (2002). Complex mathematical programming formulations of commonality and assemble-to-order systems include Dogramaci (1979), Swaminathan and Tayur (1998), Thonemann and Brandau (2000), and Lu and Song (2003). Another approach is taken by Plambeck and Ward (2003a, 2003b), who model an assemble-to-order system as a stochastic control problem. Plambeck and Ward (2003b) analyze the assemble-to-order system under the assumption that out-of-stock components are expedited at extra cost but with zero lead-time, so that 100% fill rate is achieved. With the opportunity to expedite, the multi-dimensional assemble-to-order control problem separates into single-item inventory control problems, i.e., the optimal control policy for each component is independent of all other components. The role of the secondary market in our model is somewhat similar to component expediting – both expediting and the secondary market enable the firm to satisfy all demand and, thus, significantly reduce the complexity of an otherwise very difficult problem. For a recent review of the literature related to assemble-to-order systems, see Song and Zipkin (2003).

A significant amount of the resource flexibility literature studies a conceptually similar problem of ex ante acquisition and ex post allocation of flexible and dedicated capacities (see e.g., Chod and Rudi 2005, and references therein). In fact, Van Mieghem (2004) shows the equivalence between the commonality and flexible capacity problems. A common assumption of the flexibility literature is that each product requires only a single resource (which can be flexible or dedicated). Clearly, this does not capture complex production systems where different product configurations require different combinations of multiple inputs. This paper unveils the fundamental impact that such complexity of production requirements has on the value of flexibility and how it is driven by market factors. Finally, recent treatments of the secondary market for excess inventory include Lee and Whang (2002) and Dong and Durbin (2005). These articles study excess inventory trading between n firms after (partial) resolution of demand uncertainty. In this literature, the firms trade a single product or a single component, as opposed to trading multiple components considered here.

In the next section, we formulate our base-case model and characterize the firm's optimal strategy. In §4, we consider a benchmark make-to-stock scenario and use it to quantify the benefits of production postponement. In §5, we characterize the effects of demand correlation and variability on the value of postponement and firm's profit. The optimal pricing policy is discussed in §6. In §7, we numerically examine implications of some of our assumptions. Our findings are summarized in §8. All proofs are relegated to the Appendix.

3 Model and optimal strategy

The expectation operator and the variance-covariance matrix are denoted $\mathbb{E}$ and $\mathbb{V}$, respectively. Vectors and matrices are denoted by bold lower-case and bold upper-case letters, respectively. All vectors are column vectors, and superscripts T and -1 denote transpose and inverse, respectively. The identity matrix is denoted $\mathbf{I}$. A matrix whose elements are all 0's (1's) is denoted as $\mathbf{0}$ ($\mathbf{1}$). Finally, $[m \times n]$ following a matrix denotes its dimensionality.

We consider a single period model of a firm that must choose input levels under demand uncertainty, but is able to postpone its complex production and pricing decisions until this uncertainty is resolved. The firm produces n different products using m different inputs. The input-output transformation technology is defined using an $m \times n$ technology matrix $\mathbf{A}$, where $(\mathbf{A})_{ij}$ is the amount of input i used by one unit of product j. Therefore, production of the output vector $\mathbf{y}_1$ requires the vector of input levels $\mathbf{x} = \mathbf{A}\mathbf{y}_1$. We allow the technology matrix $\mathbf{A}$ to be very general – any of the n products may use any amount of any of the m inputs.

The firm faces stochastic downward sloping demand curves for all of its products. The products are allowed to be substitutes or complements – demand for any product may depend on the price of any other product. We assume the demand-price relationship to be linear and exposed to additive uncertainty. Specifically, the output vector $\mathbf{y}_1$ demanded at the price vector $\mathbf{p}_1$ is given by

$$\mathbf{y}_1 (\boldsymbol{\xi}_1, \mathbf{p}_1) = \boldsymbol{\xi}_1 - \mathbf{D}_1\mathbf{p}_1, \tag{1}$$

where vector $\boldsymbol{\xi}_1$ represents the uncertain demand levels, and matrix $\mathbf{D}_1$ captures the demand sensitivity to prices. Specifically, the diagonal elements of $\mathbf{D}_1$ capture the own-price demand sensitivities, i.e., $(\mathbf{D}_1)_{ii} = -\partial y_i/\partial p_i > 0$, while the off-diagonal elements represent the cross-price demand sensitivities, i.e., $(\mathbf{D}_1)_{ij} = -\partial y_i/\partial p_j$, for $i \neq j$. If $(\mathbf{D}_1)_{ij} < 0$, products i and j are substitutes, and if $(\mathbf{D}_1)_{ij} > 0$, they are complements. As is standard in the economics literature, we assume that $\mathbf{D}_1$ is symmetric and positive definite and there exists a unique vector of market clearing prices $\mathbf{p}_1 = \mathbf{D}_1^{-1} (\boldsymbol{\xi}_1 - \mathbf{y}_1)$, for any output vector $\mathbf{y}_1$.[1]

As is typical in most applications, the firm must choose the vector of input levels $\mathbf{x}$, which is acquired at the unit cost vector $\mathbf{c}$, while the demand levels are uncertain. However, following the work of Lee and Whang (2002), we assume there exists a secondary market for excess inputs, where the firm can buy or sell inputs after demand uncertainty is resolved.[2] We assume that the demand (supply) curves in the secondary market are also linear and exposed to additive uncertainty, i.e., the vector of input quantities sold (purchased) in the secondary market is given by

$$\mathbf{y}_2 (\boldsymbol{\xi}_2, \mathbf{p}_2) = \boldsymbol{\xi}_2 - \mathbf{D}_2\mathbf{p}_2, \tag{2}$$

where $\boldsymbol{\xi}_2$ is the vector of secondary market demand (supply) shocks, $\mathbf{p}_2$ is the vector of secondary market input prices, and $\mathbf{D}_2$ is an invertible, symmetric and positive definite matrix of the secondary market cross-price effects. The positive (negative) elements of $\mathbf{y}_2$ correspond to input quantities sold (purchased) by the firm in the secondary market.

[1] The linear relationship between price and demand, as well as the symmetry and invertibility of $\mathbf{D}_1$ follow from the second-order approximation of the consumer utility function. Positive definiteness of $\mathbf{D}_1$ then follows from the strict concavity of this approximate utility function. Intuitively, positive definitness guarantees that the effect of a marginal price increase on the product demand is more significant than the sum of its effects on other product demands. This is needed to prevent pathological scenarios such as total demand increasing in prices.

[2] Our model is simpler than that of Lee and Whang (2002) in that we consider a single sales period.

In formulating the firm's decision problem, it is useful to think of each input as a "product" that consists of a single input. The secondary market can be then incorporated in the problem formulation by defining a new technology matrix $\mathbf{B} = (\mathbf{A}\ \mathbf{I})$, and letting $\mathbf{y}^T = \left(\mathbf{y}_1^T, \mathbf{y}_2^T\right)$, $\boldsymbol{\xi}^T = \left(\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T\right)$, $\mathbf{p}^T = \left(\mathbf{p}_1^T, \mathbf{p}_2^T\right)$ and $\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix}$. Note that our definition of $\mathbf{D}$ assumes zero cross-price effects between the end-product market and the secondary market. This, however, does not prevent these markets from being correlated. In particular, we assume that $\boldsymbol{\xi}$ follows an arbitrary probability distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. (To distinguish between the mean demand shocks in the end-product market and those in the secondary market, we simply partition $\boldsymbol{\mu}^T = \left(\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T\right)$).

If we assume constant marginal costs of production and normalized them, without loss of generality, to zero,[3] the firm's total profit can be written as

$$\pi(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^T \mathbf{c} + \mathbf{y}^T \mathbf{p}(\boldsymbol{\xi}, \mathbf{y}). \tag{3}$$

While the input acquision decision is made under demand uncertainty, the production, pricing and secondary market trading decisions can be postponed until the demand levels are observed. Production postponement means that the firm produces to order, while price postponement reflects the relative pricing flexibility. The firm's decision problem can be formulated as the following two-stage stochastic program:

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \mathbb{E}\pi(\mathbf{x}, \mathbf{y}^*), \tag{4}$$

$$\text{and } \mathbf{y}^* = \arg\max_{\mathbf{y} \in S(\mathbf{x})} \pi(\mathbf{x}, \mathbf{y}), \tag{5}$$

where $S(\mathbf{x})$ is the set of feasible output vectors. In the first stage, i.e., while the demand conditions in terms of $\boldsymbol{\xi}$ are uncertain, the firm decides the input vector $\mathbf{x}$. In the second stage, i.e., after the demand conditions are observed, the firm sets end-product price vector $\mathbf{p}_1$ and produces to order the end-product output vector $\mathbf{y}_1(\boldsymbol{\xi}_1, \mathbf{p}_1)$. At the same time, the firm trades input vector $\mathbf{y}_2$ at the secondary market clearing price vector $\mathbf{p}_2(\boldsymbol{\xi}_2, \mathbf{y}_2)$. Because of the one-to-one relationship between the price and demand vectors, we can let the second-stage decision variable be the total "output vector" $\mathbf{y}$, which is sold at the price vector $\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}) = \mathbf{D}^{-1}(\boldsymbol{\xi} - \mathbf{y})$.

For the model specification to be complete, it remains to characterize the set of feasible output vectors $S(\mathbf{x})$. Clearly, the output of each end-product must be non-negative, i.e., $\mathbf{y}_1 \geq \mathbf{0}$. Rather then imposing non-negativity constraints on $\mathbf{y}_1$, we follow the standard economics literature[4] in

[3] One can simply think of each end-product price as price minus the marginal costs of production.

[4] See, for example, Vives (1984), p. 77; or Fershtman and Judd (1987), p. 935.

assuming that the support of the demand shock distribution is appropriately bounded so that the unconstrained output of each end-product is always non-negative. In §7.2, we characterize these bounds precisely (Lemma 4) and illustrate graphically how restrictive they are (Figure 3).

The output vector is also constrained by the input availability, i.e., $\mathbf{By} \leq \mathbf{x}$. For the sake of analytical tractability, we consider the "clearance" strategy introduced by Van Mieghem and Dada (1999) and, in the risk-pooling context, by Chod and Rudi (2005). Input clearance means that the firm always "clears" all of its inputs, either uses them in production of end-products or sells them in the secondary market. The output feasibility set thus becomes

$$S(\mathbf{x}) = \{\mathbf{y} : \mathbf{By} = \mathbf{x}\}.$$

It is possible that, under extremely low realizations of the secondary market demand shocks, the firm would maximize its revenue by withholding some of the excess input inventory from the secondary market to prevent the secondary market price from dropping too much. In §7.2, we show that if the support of the demand shock distribution is appropriately bounded, this is a zero probability event. In general, we believe this to be a very low probability event, in particular when the firm is only one of many players in the secondary market and thus has a small impact on the price.[5] In practice, input clearance may also result from phenomena outside the scope of this paper such a manager's reluctance to scrap inventory. Finally, if we interpret the secondary market in a more abstract way as a mechanism that determines the salvage value of the firm's unused inputs, input clearance simply means that the unit salvage value of each input decreases in the excess amount at a *linear* rate.[6]

To characterize the optimal solution, we will need the following result.

Lemma 1 *The matrix* $\begin{pmatrix} 2\mathbf{I} & \mathbf{DB}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ *is invertible.*

Before we characterize the optimal input and output vectors and the firm's expected profit in the next proposition, it is useful to define $\mathbf{Q}$ as the $(n+m) \times (n+m)$ upper-left submatrix of $\begin{pmatrix} 2\mathbf{I} & \mathbf{DB}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}^{-1}$.

[5] Our assumption of downward sloping demand curves implies that the firm has some impact on each price. This impact, however, can be arbitrarily small.

[6] Chod and Rudi (2005) show for the case of $\mathbf{A} = (1\ 1)$ and $\mathbf{D} = \mathbf{I}$ that even under normally distributed demand shocks when clearance is not necessarily the optimal decision rule, considering the clearance strategy does not alter the directional effects of demand variability and correlation on the firm's profit.

Proposition 1 *The optimal output vector, its expected value and covariance matrix are, respectively,*

$$\mathbf{y}^* = \mathbb{E}\mathbf{y}^* + \mathbf{Q}\left(\boldsymbol{\xi} - \boldsymbol{\mu}\right), \quad \mathbb{E}\mathbf{y}^* = \frac{1}{2}\left(\boldsymbol{\mu} - \mathbf{D}\mathbf{B}^T\mathbf{c}\right) \quad \textit{and} \quad \mathbb{V}\mathbf{y}^* = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T.$$

The optimal input vector and the corresponding expected profit are, respectively,

$$\mathbf{x}^* = \mathbf{B}\mathbb{E}\mathbf{y}^* \quad \textit{and} \quad \mathbb{E}\pi\left(\mathbf{x}^*, \mathbf{y}^*\right) = \mathbb{E}\mathbf{y}^{*T}\mathbf{D}^{-1}\mathbb{E}\mathbf{y}^* + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\left(\mathbf{D}^{-1}\right)_{ij}\left(\mathbb{V}\mathbf{y}^*\right)_{ij}.$$

To ensure non-negativity of the optimal input vector $\mathbf{x}^*$, we need to assume $\mathbf{B}\boldsymbol{\mu} \geq \mathbf{B}\mathbf{D}\mathbf{B}^T\mathbf{c}$. This rules out the pathological scenario in which it is optimal to ex ante "short sell" some inputs in the hope of profiting from their lower secondary market prices. This, however, does not mean that the firm cannot buy less (or more) inputs than it expects to use for its own production. Consider, for the ease of exposition, the simplest form of the secondary market price-sensitivity matrix, i.e., $\mathbf{D}_2 = \mathbf{I}$. If $\mu_{n+i} < (>)\, c_i$, the expected secondary market price of input i is lower (higher) than its ex-ante cost, i.e., $\mathbb{E}p_{n+i}(\boldsymbol{\xi}, \mathbf{y}^*) < (>)\, c_i$, and the firm expects to buy (sell) this input in the secondary market taking advantage of price arbitrage, i.e., $\mathbb{E}y^*_{n+i} < (>)\, 0$. Only if $\mu_{n+i} = c_i$, then the expected secondary market price of input i equals its ex-ante cost, i.e., $\mathbb{E}p_{n+i}(\boldsymbol{\xi}, \mathbf{y}^*) = c_i$, the expected quantity of input i sold in (purchased from) the secondary market is zero, i.e., $\mathbb{E}y^*_{n+i} = 0$.

4 Value of production postponement

To gain more insight into the expected profit expression, it is useful to consider a benchmark "make-to-stock" scenario in which the output decision must be made *before* demand uncertainty is resolved. For consistency with the make-to-order scenario, we assume that the firm always sells its entire end-product output and any unused inputs at the market clearing prices.[7] When deciding the optimal output vector, the firm faces the following optimization problem:

$$\hat{\mathbf{y}} = \arg\max_{\mathbf{y}} \mathbb{E}\pi\left(\mathbf{B}\mathbf{y}, \mathbf{y}\right). \tag{6}$$

The input vector is then implied by the minimum production requirements, i.e., $\mathbf{x} = \mathbf{B}\mathbf{y}.$ We characterize the optimal input and output vectors as well as the expected profit in the make-to-stock scenario in Lemma 2.

Lemma 2 *The optimal output vector, input vector and the corresponding expected profit in the make-to-stock scenario are, respectively*

$$\hat{\mathbf{y}} = \mathbb{E}\mathbf{y}^*, \quad \hat{\mathbf{x}} = \mathbf{B}\hat{\mathbf{y}}, \quad \textit{and} \quad \mathbb{E}\pi\left(\hat{\mathbf{x}}, \hat{\mathbf{y}}\right) = \mathbb{E}\mathbf{y}^{*T}\mathbf{D}^{-1}\mathbb{E}\mathbf{y}^*. \tag{7}$$

[7] Note that the firm may also buy additional inputs ex post.

Note that the expected profit in the make-to-stock scenario is affected by neither demand variability nor demand correlation.[8] This is a direct consequence of our clearance assumption: the realized demand levels affect the output prices but not the output itself. Demand variability and correlation thus influence the profit distribution but not its expected value. This is not the case in our base-case make-to-order scenario where demand variability and correlation affect the expected profit by affecting the *value of production postponement*. We characterize this value in Corollary 1.

Corollary 1 *The value of production postponement is*

$$V = \mathbb{E}\pi\left(\mathbf{x}^*, \mathbf{y}^*\right) - \mathbb{E}\pi\left(\hat{\mathbf{x}}, \hat{\mathbf{y}}\right) = \sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\left(\mathbf{D}^{-1}\right)_{ij}\left(\mathbb{V}\mathbf{y}^*\right)_{ij}. \tag{8}$$

The value of production postponement V represents the benefit of having perfect demand information at the time of making the output decision. Although Corollary 1 together with Proposition 1 enable us to calculate the value of postponement for any technology matrix $\mathbf{A}$, price sensitivity matrix $\mathbf{D}$ and covariance matrix $\mathbf{\Sigma}$, its further analytical treatment at this level of generality is difficult. We thus focus on a particular form of the technology matrix that is characteristic of mass-customized assembly-based products. In particular, we consider a product (e.g., personal computer) that consists of k components (hard disc, processor, memory, etc.), each of which exists in l different versions (e.g., hard disc can have capacity of 20, 40, 60 or 80 GB).[9] Therefore, the firm offers $n = l^k$ end-product configurations that are assembled from $m = kl$ different components. An example of such product configuration with $k = 2$ and $l = 2$ is illustrated in Figure 1.

Recall that the relationship between an end-product and a component is captured by the technology matrix $\mathbf{A}$ so that $(\mathbf{A})_{ij} = 1$ if end-product j uses component i, and $(\mathbf{A})_{ij} = 0$ otherwise. To capture the relationship between two end-products, we define an $n \times n$ "commonality matrix" $\mathbf{C}$ so that $(\mathbf{C})_{ij}$ represents the number of components shared by end-products i and j. Finally, to characterize the relationship between two components, we define an $m \times m$ "technological complementarity matrix" $\mathbf{Z}$ so that $(\mathbf{Z})_{ij} = 1$ if components i and j are technological complements such as a hard disc and a processor, and $(\mathbf{Z})_{ij} = 0$ if components i and j are technological substitutes such as a 20GB hard disc and an 80GB hard disc.

[8] We use terms "demand variability" and "demand correlation" for the variability and correlation of the demand shock vector $\boldsymbol{\xi}$.

[9] The assumption of the same variety in each component class is made without loss of generality. In particular, one can introduce the appropriate number of "virtual components" for each component class to bring the variety in this class to some common l. To ensure that the output of all products that involve some "virtual components" is set to zero, we let $\mu_i = 0$, $\sigma_i = 0$, $(\mathbf{D})_{ij} = 0$ for all $j \neq i$ and $(\mathbf{D})_{ii} \longrightarrow 0$ for all i's that contain "virtual components." This results in $\mathbb{E}y_i^* \longrightarrow 0$ and $\mathbb{V}y_i^* \longrightarrow 0$, so that "virtual products" are not produced.

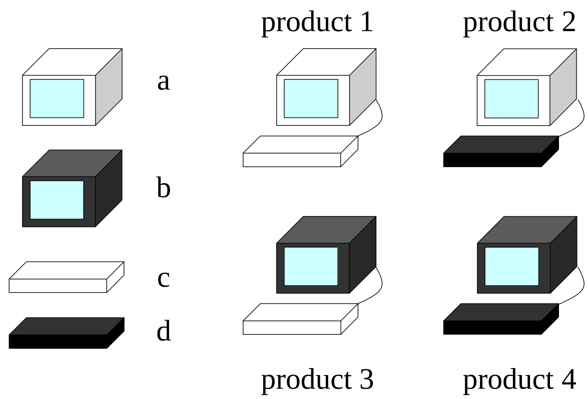

Figure 1: An example of the product that involves 2 components, each available in 2 versions.

We first examine the value of production postponement analytically for the special case of identical own-price sensitivities and zero cross-price sensitivities of demands, i.e., $\mathbf{D} = \mathbf{I}$. In §7.1, we examine the robustness of our analytical results numerically under more general price-demand relationships. The next lemma is critical to our analysis.

Lemma 3 *If* $\mathbf{D} = \mathbf{I}$*, the value of production postponement can be written as*

$$
\begin{aligned}
V &= \sum_{i=1}^{n+m} \left(\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T\right)_{ii}, \textit{ where } \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix} \textit{ and} \\
\mathbf{Q}_1\,[n \times n] &= \frac{\left(k^2 - k\right) l^{k-2}}{2\left(1 + l^{k-1}\right)\left(1 + k l^{k-1}\right)}\mathbf{1} - \frac{1}{2\left(1 + l^{k-1}\right)}\mathbf{C} + \frac{1}{2}\mathbf{I}, \\
\mathbf{Q}_2\,[n \times m] &= \frac{(k-1)\, l^{k-2}}{2\left(1 + l^{k-1}\right)\left(1 + k l^{k-1}\right)}\mathbf{1} - \frac{1}{2\left(1 + l^{k-1}\right)}\mathbf{A}^T, \\
\mathbf{Q}_3\,[m \times n] &= \frac{(k-1)\, l^{k-2}}{2\left(1 + l^{k-1}\right)\left(1 + k l^{k-1}\right)}\mathbf{1} - \frac{1}{2\left(1 + l^{k-1}\right)}\mathbf{A}, \\
\mathbf{Q}_4\,[m \times m] &= \frac{-(k-1)\, l^{2k-3}}{2\left(1 + l^{k-1}\right)\left(1 + k l^{k-1}\right)}\mathbf{1} + \frac{l^{k-2}}{2\left(1 + l^{k-1}\right)}\mathbf{Z} + \frac{l^{k-1}}{2\left(1 + l^{k-1}\right)}\mathbf{I}.
\end{aligned}
$$

In the next section, we discuss how the value of production postponement and therefore the firm's expected profit depend on demand correlation and variability.

5 Effects of demand correlation and variability

There are three types of correlation: (i) correlation between end-product demands, (ii) correlation between an end-product demand and a component demand and (iii) correlation between component demands. We characterize the effect of each of these correlations in the following three propositions, respectively.

Proposition 2 *If* $\mathbf{D} = \mathbf{I}$*, the expected profit increases in demand correlation between end-products* u *and* v *if and only if the number of components shared by these products does not exceed a threshold* $\tau_0 = \frac{k(k-1)l^{k-2}}{1+kl^{k-1}}$*:*

$$\frac{\partial \mathbb{E}\pi\left(\mathbf{x}^*, \mathbf{y}^*\right)}{\partial \rho_{uv}} = \frac{\partial V}{\partial \rho_{uv}} > 0 \qquad \Longleftrightarrow \qquad (\mathbf{C})_{uv} < \tau_0.$$

Proposition 3 *If* $\mathbf{D} = \mathbf{I}$*, the expected profit increases in demand correlation between end-product* u *and component* v *if and only if the end-product does not contain the component:*

$$\frac{\partial \mathbb{E}\pi\left(\mathbf{x}^*, \mathbf{y}^*\right)}{\partial \rho_{u,n+v}} = \frac{\partial V}{\partial \rho_{u,n+v}} > 0 \qquad \Longleftrightarrow \qquad (\mathbf{A})_{vu} = 0.$$

Proposition 4 *If* $\mathbf{D} = \mathbf{I}$*, the expected profit increases in demand correlation between components* u *and* v *if and only if the components are technological complements:*

$$\frac{\partial \mathbb{E}\pi\left(\mathbf{x}^*, \mathbf{y}^*\right)}{\partial \rho_{n+u,n+v}} = \frac{\partial V}{\partial \rho_{n+u,n+v}} > 0 \qquad \Longleftrightarrow \qquad (\mathbf{Z})_{uv} = 1.$$

To develop more intuition for these results, consider the product illustrated in Figure 1. In this example, any two products either share one component or do not share any components. Higher demand correlation between products that share a component results in higher variability of the usage of this component and, in line with intuition, reduces the efficacy of risk pooling and the value of postponement. The effect of demand correlation between products that do not share any component is less intuitive. Suppose demand for product 2 is relatively high. The high demand for product 2 results in a high usage of its components a and d, limiting the output of products 1 and 4, which also use these two components. The limited output of products 1 and 4 then results in a relative abundance of components b and c, which together constitute product 3. This is the situation when high demand for product 3 is particularly desirable. In other words, risk pooling works best if demands for products 2 and 3, which do not share any common components, are positively correlated. Thus, in this example, the value of postponement increases in product demand correlation if and only if the two products do not have a commonality. In general, the value of postponement increases in product demand correlation if and only if the two products have *relatively few* commonalities.

To understand the effect of the correlation between a product demand and a component demand, suppose again that demand for product 2 is relatively high. Since components a and d used in product 2 are likely to be *purchased* by the firm from the secondary market, it is desirable that the secondary market demand (and hence price) for these components is relatively low. Furthermore, the relatively high usage of components a and d limits the output of products 1 and 4, which is likely to result in an excess inventory of components b and c. Thus, components b and c, which are not used in product 2, are likely to be *sold* in the secondary market and, thus, their high demand (price) in this market is desirable. In general, the value of postponement increases in the correlation between demand for a product and demand for a component if and only if the product does not contain this component.

Finally, to build some intuition for the effect of correlation between component demands, suppose there is relatively high secondary market demand for component a. The resulting high price of this component is an incentive for the firm to *sell* this component in the secondary market. This limits the output of products 1 and 2 that use component a, resulting in a relative excess of technologically complementary components c and d. Since these components are then likely to be *sold* in the secondary market, it is desirable that they are highly demanded (priced) there. The relatively scarce component a can be also "substituted" with component b, which is therefore likely to be purchased in the secondary market. It is thus desirable that the secondary market demand (price) of component b is low. In general, the value of postponement increases in the correlation between component demands if and only if the components are technological complements.

We have shown that demand correlation between products 1 and N matters even if these products do not share any inputs, as long as product 1 shares an input with product 2 that shares an input with product 3, and so on, up to product $N-1$ that shares an input with product N. Such "chaining" of commonalities has a similarity with the "ripple effect" identified in the seminal work of Fine and Freund (1990) who model the trade-off between flexibility and acquisition cost of three types of capacity: capacity A (B) is dedicated to product A (B) while capacity AB can be used to make both products. The ripple effect means that an increase in the cost of capacity A results in its substitution with capacity AB, which in turn leads to a decrease in the optimal level of capacity B.

The next two propositions characterize the impact of variability of end-product demand and component demand, respectively.

Proposition 5 *If* $\mathbf{D}=\mathbf{I}$*, the expected profit increases in the demand variability of end-product* u*, i.e.,* $\frac{\partial \mathbb{E}\pi(\mathbf{x}^*,\mathbf{y}^*)}{\partial \sigma_u}=\frac{\partial V}{\partial \sigma_u}>0$*, if and only if*

$$\tau_1\sigma_u^2+\sum_{\substack{i=1,i\neq u\\ (\mathbf{C})_{ui}<\tau_0}}^{n}\sigma_{iu}\left[\tau_0-(\mathbf{C})_{ui}\right]+\tau_2\sum_{\substack{j=1\\ (\mathbf{A})_{ju}=0}}^{m}\sigma_{n+j,u}>\sum_{\substack{i=1,i\neq u\\ (\mathbf{C})_{ui}>\tau_0}}^{n}\sigma_{iu}\left[(\mathbf{C})_{ui}-\tau_0\right]+\tau_3\sum_{\substack{j=1\\ (\mathbf{A})_{ju}=1}}^{m}\sigma_{n+j,u},$$

where $\tau_1 = \tau_0 + 1 + l^{k-1} - k > 0$, $\tau_2 = \tau_0/k > 0$ *and* $\tau_3 = 1 - \tau_0/k > 0$.

Proposition 6 *If* $\mathbf{D} = \mathbf{I}$*, the expected profit increases in the demand variability of component* v*, i.e.,* $\frac{\partial \mathbb{E}\pi(\mathbf{x}^*,\mathbf{y}^*)}{\partial \sigma_{n+v}} = \frac{\partial V}{\partial \sigma_{n+v}} > 0$*, if and only if*

$$\sigma_{n+v}^2 + \tau_4 \sum_{\substack{j=1 \\ (\mathbf{Z})_{jv}=1}}^{m} \sigma_{n+v,n+j} + \tau_5 \sum_{\substack{i=1 \\ (\mathbf{A})_{vi}=0}}^{n} \sigma_{n+v,i} > \tau_6 \sum_{\substack{j=1,\ j\neq v \\ (\mathbf{Z})_{jv}=0}}^{m} \sigma_{n+v,n+j} + l^{1-k} \sum_{\substack{i=1 \\ (\mathbf{A})_{vi}=1}}^{n} \sigma_{n+v,i},$$

where $\tau_4 = \frac{l^{k-1}+1}{l+l^{k-1}+kl^k-kl^{k-1}} > 0$, $\tau_5 = \frac{k-1}{l+l^{k-1}+kl^k-kl^{k-1}} > 0$ *and* $\tau_6 = \frac{(k-1)l^{k-1}}{l+l^{k-1}+kl^k-kl^{k-1}} > 0$.

Demand variability consists of the variance of a particular demand shock and of its covariances with other demand shocks. All demand variances have a positive impact on the value of postponement because they increase the option value of risk-pooling. The impact of demand covariances on the value of postponement goes hand in hand with the impact of the corresponding correlations. In particular, the following covariances have a positive impact on the value of postponement: (i) covariance between demands for products with relatively few commonalities, (ii) covariance between demand for a product and demand for a component that is not used in this product and (iii) covariance between demands for components that are technological complements. The covariances with a negative impact on the value of postponement include: (i) covariance between demands for products with a relatively high number of commonalities, (ii) covariance between demand for a product and demand for a components that is used in this product and (iii) covariance between demands for components that are technological substitutes. Propositions 5 and 6 state that the value of postponement increases in demand variability if, and only if, the components of this variability with a positive impact on the value of postponement exceed those with a negative impact.

The link between the effect of demand correlation and input commonality has also implications for the optimal pricing policy which we discuss next.

6 Optimal prices

We have established that risk-pooling is most effective when demands for products with a high (low) degree of commonality move in the opposite (the same) direction. Since demand is a function of price, the commonality structure of the product line should be reflected in the pricing decision. In particular, we would expect the optimal price vector to increase (decrease) the correlation between demands for products characterized by a low (high) degree of commonality. Because pricing is endogenous in our model, we can verify our intuition. To isolate the effect that commonality has on the price correlation, we assume that all demand shocks are independent. For simplicity, we also assume that demands have the same standard deviations.

Proposition 7 *If* $\mathbf{D} = \mathbf{I}$ *and* $\mathbf{\Sigma} = \mathbf{I}$*, the optimal prices of end-products* u *and* v *are positively correlated if and only if the number of components shared by these products exceeds a threshold:*

$$(\mathbb{V}\mathbf{p}\,(\boldsymbol{\xi}, \mathbf{y}^*))_{uv} > 0 \qquad \Longleftrightarrow \qquad (\mathbf{C})_{uv} > \tau_0.$$

Proposition 7 states that the optimal prices of two products are positively (negatively) correlated if and only if the products are characterized by a high (low) degree of commonality. This may seem counterintuitive. Since we want demands for products with a high (low) degree of commonality to be negatively (positively) correlated, one may argue that the firm should negatively (positively) correlate their prices. The optimal pricing policy is more subtle: When the level of demand for product u is high, the optimal price of product u is also high. The large output of product u (and the resulting large usage of components in its bill of material) makes it more economical to produce products of low commonality with product u than products of high commonality with product u. To induce the desired demand levels, the firm needs to set relatively low prices of products of low commonality with product u and relatively high prices of products of high commonality with product u. In other words, the optimal prices of products with high (low) commonality will tend to move in the same (opposite) direction, as stated in Proposition 7. Note that Proposition 7 captures only one factor of price correlation which is input commonality. In reality, price correlation will depend also on demand shock correlation and cross-price effects. A numerical investigation confirmed our intuition that positive demand shock correlation and product substitutability increase price correlation.

7 Discussion

In this section, we scrutinize two of our assumptions. Recall that Propositions 2-6 are based on a highly stylized price-demand relationship when $\mathbf{D} = \mathbf{I}$. In §7.1, we examine robustness of these Propositions 2-6 under more general price sensitivity matrices. Then, in §7.2, we characterize demand shock distributions that guarantee non-negativity of end-product output vector.

7.1 General price sensitivity of demand

Before examining price effects in the end-product market, we focus on the impact of own-price sensitivity of the secondary market demand.[10] Our measure of own-price sensitivity of secondary market demands, i.e., the slope of the secondary market demand functions, depends, among other factors such as the firm's size relative to the market, component standardization, etc., on the cost of the component. Presumably, a dollar increase in the price of a component worth a few dollars will have significantly greater impact on the quantity demanded than a dollar increase in the price of a component worth several hundred dollars, i.e., if $c_i \gg c_j$, we expect $(\mathbf{D})_{n+i,n+i} < (\mathbf{D})_{n+j,n+j}$. A lower own-price sensitivity of component demand in terms of $(\mathbf{D})_{n+i,n+i}$ means that selling (buying) this component in the secondary market results in a sharper decrease (increase) of the component price, making this component more critical from the inventory management perspective.

To gain more insight into the effect of unequal own-price sensitivities of component demands, consider a product consisting of one expensive and several relatively inexpensive components. Suppose, furthermore, that $(\mathbf{D})_{ii} = 0.1$ for all versions of the expensive component, and $(\mathbf{D})_{ii} = 1$ for all of the inexpensive components. We can transform this problem into an equivalent problem in which $(\mathbf{D})_{ii} = 1$ for all $i > n$, by replacing the expensive component with a bundle of 10 inexpensive "virtual components," whose demand shocks are equal to the demand shock of the original component but whose demands are 10 times more sensitive to own price than the demand of the original component. This choice of parameters ensures that the total price of the 10 "virtual components" is always equal to the price of the original component $p_i(\xi_i, y_i) = 10(\xi_i - y_i)$.[11] Therefore, sharing a component with own-price sensitivity of demand $(\mathbf{D})_{ii} = 0.1$ is equivalent to sharing 10 components with own-price sensitivity of demand $(\mathbf{D})_{ii} = 1$. In general, under unequal own-price sensitivities of secondary market demands, the degree of commonality between two products cannot be measured by the sheer number of common components. Instead, each component needs to be weighted by the reciprocal value of its secondary market demand sensitivity to own price, which is strongly related to its cost.

To examine robustness of our results under more general price sensitivity of the end-product demands, we run an extensive numerical analysis, the results of which are illustrated in Figure 2.

[10] We do not examine cross-price effects in the secondary market as we expect them to be negligible.

[11] For the two problem formulations to be truly equivalent, one has to also ensure that products containing a "mix" of different versions of an original component are not produced in the new formulation. As we discussed in Footnote 9, a zero output of any product can be ensured by the appropriate choice of parameters. However, by letting $(\mathbf{D})_{ii} \longrightarrow 0$, we depart from our assumption that all end-products have the same own-price demand sensitivity.

Since demand variability and correlation affect the expected profit only as far as they affect the value of production postponement, we only plot this value. All graphs in this figure are based on a product consisting of two components, each of which exists in two versions (see Figure 1 for illustration) and the following base-case parameter values. The cost vector $\mathbf{c}^T = (1.5, 1, 0.75, 0.5)$ indicates that type 1 component is more expensive than type 2 component and both types of components exist in one high-end and one low-end version. The price-sensitivity matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} \text{ where } \mathbf{D}_1 = \begin{pmatrix} 1 & -0.1 & -0.1 & -0.1 \\ -0.1 & 1.1 & -0.1 & -0.1 \\ -0.1 & -0.1 & 1.2 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1.3 \end{pmatrix} \text{ and } \mathbf{D}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.3 \end{pmatrix},$$

indicates the following: (i) A higher-end product (a more expensive component) has a lower own-price sensitivity of demand. (ii) Products are relatively strong substitutes: 30% of customers that decide not to buy configuration 1 due to its price increase will switch to another configuration. (iii) There are no cross-price effects in the secondary market. The mean demand shock vector $\boldsymbol{\mu}$ is set so that the expected profit margin on each end-product is 10% of its component cost, i.e., $\mathbb{E}\mathbf{p}_1(\boldsymbol{\xi}_1, \mathbf{y}_1^*) - \mathbf{A}^T\mathbf{c} = 0.1 \times \mathbf{A}^T\mathbf{c}$ and, furthermore, the expected secondary market price for each component equals to the original cost of this component, i.e., $\mathbb{E}\mathbf{p}_2(\boldsymbol{\xi}_2, \mathbf{y}_2^*) = \mathbf{c}$. The standard deviation of each demand shock is set so that its coefficient of variation $\sigma_i/\mu_i = 0.1$, $i = 1, ..., n+m$. (In the base case, this variability of the demand shocks results in the coefficient of variation of demands $\sqrt{\mathbb{V}y_i^*}/\mathbb{E}y_i^* \approx 0.3$.) Finally, we assume that all end-products have positively correlated demands with $\rho = 0.5$. Similarly, demands for components that are technological substitutes are positively correlated with $\rho = 0.5$. There is no correlation between demand for end-products and demand for components or between demands for components that are technological complements. Note that whenever we vary some entries of the price sensitivity matrix $\mathbf{D}$ (i.e., some of the slopes of the demand curves), we also adjust the mean demand shocks $\boldsymbol{\mu}$ (the intercepts of the demand curves) and their standard deviations $\boldsymbol{\sigma}$ to keep the expected profit margins and the coefficients of variation of $\boldsymbol{\xi}$ unchanged.

Figures 2a-c show the effect of demand variability and correlation on the value of postponement under different degrees of substitutability/complementarity of end-products: the cross-price sensitivity of end-product demand $(\mathbf{D})_{ij}$, $i, j \leq n, i \neq j$, is varied from -0.2 (strong substitutes) to 0.2 (strong complements). Figures 2d-f plot the effect of demand variability and correlation under different own-price sensitivities of the secondary market demands: their vector $((\mathbf{D})_{55}, (\mathbf{D})_{66}, (\mathbf{D})_{77}, (\mathbf{D})_{88})$ is varied from $(0.1, 0.11, 0.12, 0.13)$ (very inelastic secondary market

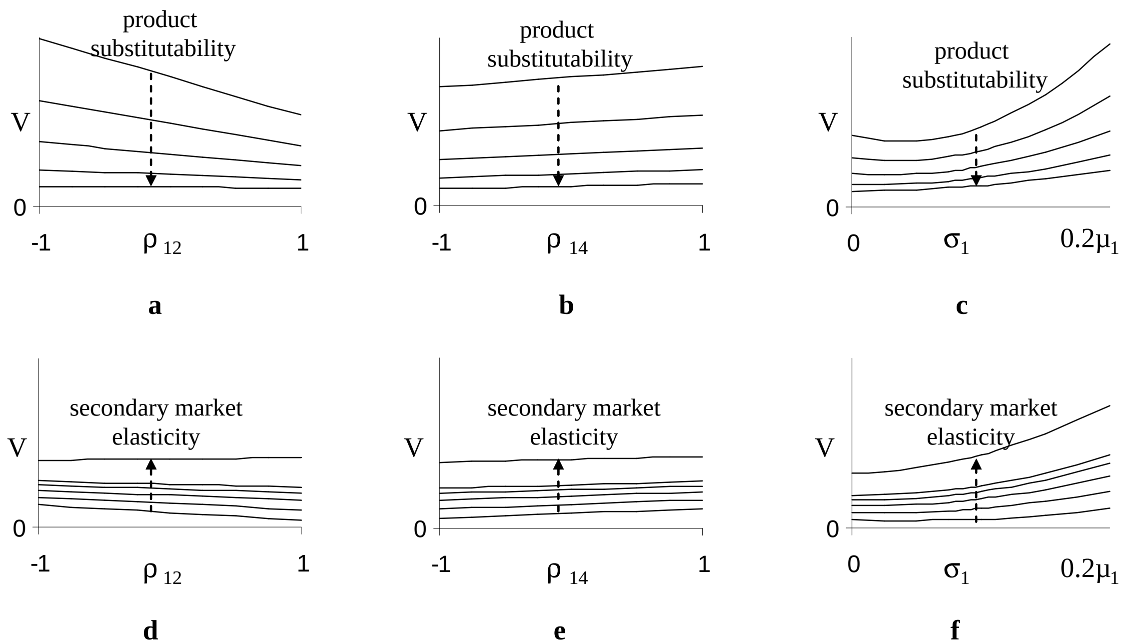

Figure 2: The effect of demand correlation and variability on the value of postponement for different price-sensitivities of demand.

demands) to $(10, 11, 12, 13)$ (highly elastic secondary market demands).

Figures 2a and 2d show the effect of demand correlation between end-products with no commonality $\rho_{12} \in [-1, 1]$, with all other end-product demands being independent. Figures 2b and 2e show the effect of demand correlation between end-products with a commonality $\rho_{14} \in [-1, 1]$, with all other end-product demands being independent. Finally, Figures 2c and 2f show the effect of demand variability $\sigma_1 \in [0, 0.2\mu_1]$, with the coefficients of variation of all other demand shocks being fixed at 0.1.

Our numerical investigation confirmed all of the insights obtained analytically for $\mathbf{D} = \mathbf{I}$ with one exception. When products are very strong substitutes and the secondary market demands are highly elastic, the value of postponement slightly increases in ρ_{12}, the demand correlation between products that share a common component (see Figure 2d). When products are strong substitutes, the total demand shock is more important than demand shocks for individual products. Higher demand correlation means higher variability of the total demand which increases the option value of postponement. When the secondary market demand is highly elastic, i.e., the penalty for using the secondary market is rather low, this "total variability" effect of demand correlation dominates its adverse effect on risk-pooling. Note that this observation does not counter our key result that positive demand correlation increases the value of postponement as long as the products have a relatively low degree of commonality. Instead, it shows that positive demand correlation may increase the value of postponement even if the products have a relatively high degree of commonality, provided that they are substitutes and rely on components that can be traded ex-post at prices that are fairly insensitive to the quantities traded.

7.2 Output non-negativity

Recall that the end-product outputs characterized in Proposition 1 are not guaranteed to be non-negative unless we "appropriately bound" the support of the demand shock distribution. Before making this statement precise in Lemma 4, it is useful to partition matrix $\mathbf{Q}$ as $\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix}$ where $\mathbf{Q}_1$ is $n \times n$.

Lemma 4 *The vector of optimal end-product outputs $\mathbf{y}_1^*$ is guaranteed to be non-negative if and only if*

$$(\mathbf{Q}_1\ \mathbf{Q}_2)(\boldsymbol{\mu} - \boldsymbol{\xi}) \leq \frac{1}{2}\left(\boldsymbol{\mu}_1 - \mathbf{D}_1 \mathbf{A}^T \mathbf{c}\right) \text{ for any } \boldsymbol{\xi}. \tag{9}$$

First note that the right-hand side of (9) is equal to the expected optimal end-product output vector $\mathbb{E}\mathbf{y}_1^*$. A rather weak assumption of positive *expected* end-product outputs thus ensures that the right-hand side of (9) is positive and therefore (9) holds for $\boldsymbol{\xi} = \boldsymbol{\mu}$. The continuity of the left-hand side of (9) in $\boldsymbol{\xi}$ then implies that (9) will hold as long as $\boldsymbol{\xi}$ is sufficiently close to $\boldsymbol{\mu}$. What is "sufficiently close" depends on problem parameters. The lower the input cost $\mathbf{c}$ relative to the mean demand (price) shock $\boldsymbol{\mu}_1$, the higher the expected end-product output levels $\mathbb{E}\mathbf{y}_1^* = \frac{1}{2}\left(\boldsymbol{\mu}_1 - \mathbf{D}_1 \mathbf{A}^T \mathbf{c}\right)$, and the less restrictive the conditions that (9) imposes on the distribution of $\boldsymbol{\xi}$.

To gain further insight into how restrictive condition (9) really is, we consider a product consisting of two components, each of which exists in two versions as illustrated in Figure 1, with symmetrical mean demand shocks ($\mu_1 = \mu_2 = \mu_3 = \mu_4 \equiv \mu$), symmetrical component costs ($c_1 = c_2 = c_3 = c_4 \equiv c$), and no cross-price effects ($\mathbf{D} = \mathbf{I}$). Even in this simple example, the permissible support of $\boldsymbol{\xi}$ is an 8-dimensional set and thus difficult to illustrate graphically. Thus, we only graph a set of permissible ξ_1 and ξ_2 in the case when all other demand shocks are equal to their means. If $\xi_i = \mu_i$ for $i = 3, ..., 8$, condition (9) becomes

$$\max\left(\frac{30}{7}c - \frac{11}{7}\mu + \frac{3}{7}\xi_1, 15c - 8\mu + \frac{3}{2}\xi_1\right) \leq \xi_2 \leq \min\left(-10c + \frac{11}{3}\mu + \frac{7}{3}\xi_1, -10c + \frac{16}{3}\mu + \frac{2}{3}\xi_1\right). \tag{10}$$

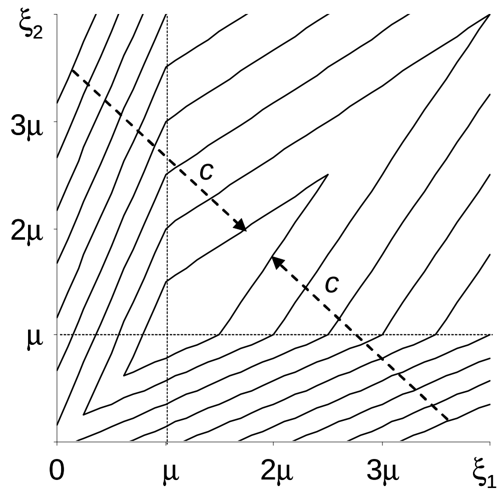

Figure 3: The support of demand shocks ξ_1 and ξ_2 that guarantees non-negative end-product outputs.

Figure 3 illustrates the set of permissible realization of ξ_1 and ξ_2 given by (10) as a function of the component cost $c \in \{0.05\mu, ..., 0.45\mu\}$. Recall that higher component costs mean lower expected output levels, making the permissible support of $\boldsymbol{\xi}$ more restricted. The worst case scenario (smallest permissible support of ξ_1 and ξ_2) illustrated in Figure 3 corresponds to $c = .45\mu$ in which case the material cost represents 90% of the expected price at which the *first unit* of the product can be sold, i.e., $2c = 0.9\mathbb{E}p_i\left(\xi_i, y_i = 0\right), i = 1, 2$. Although in this extreme case, the permissible support of ξ_1 and ξ_2 is rather small, as the component cost takes more realistic values, the permissible support of ξ_1 and ξ_2 expands very quickly. Also note that the constraints are least restrictive when the demand shocks are positively correlated.

Finally, even if the support of $\boldsymbol{\xi}$ is unbounded (such as in the case of normally distributed $\boldsymbol{\xi}$), the effect of relaxing the output non-negativity constraint will be small unless the variability of $\boldsymbol{\xi}$ or the component costs are extremely high. Chod and Rudi (2005) demonstrate, assuming a normally distributed $\boldsymbol{\xi}$, $\mathbf{A} = (1\ 1)$, and $\mathbf{D} = \mathbf{I}$, that as long as the coefficient of variation of $\boldsymbol{\xi}$ is below 0.5 and $c \leq 0.5\mu_1 = 0.5\mu_2$, relaxing the output non-negativity constraint increases expected profit by less than 1% for independent or positively correlated demands and by less than 3% if the correlation coefficient of demand equals -0.5.

In a similar way, one can restrict the support of $\boldsymbol{\xi}$ to guarantee that input clearance is always the optimal decision rule. The vector of marginal revenues from selling inputs in the secondary market is $\mathbf{D}_2^{-1}\left(\boldsymbol{\xi}_2 - 2\mathbf{y}_2^*\right)$. If $\boldsymbol{\xi} = \boldsymbol{\mu}$, the marginal revenue vector is positive and input clearance is thus optimal. By continuity, the marginal revenue vector will be positive, i.e., input clearance will be optimal, as long as $\boldsymbol{\xi}$ is sufficiently close to $\boldsymbol{\mu}$.

8 Conclusion

This paper studies resource investment, allocation and product pricing while considering a very general input-output transformation technology that is characteristic for mass customization in assembly-based industries such as computer or car manufacturing. Our contribution is threefold. First, we contribute to the operations management literature by challenging the commonly accepted insight that the benefits of risk-pooling are always increasing in demand variability and decreasing in the correlation between demands for products that rely on a common resource. We show that the effects of both demand variability and demand correlation depend critically on the commonality structure of the entire product line. For example, we prove that the benefits of risk-pooling increase in demand correlation if and only if the degree of commonality between the products does not exceed a threshold.

Second, we contribute to the revenue management literature by showing how product pricing and other demand-influencing levers should be used to maximize the benefits of risk-pooling. In particular, we show that a high degree of commonality between two products is a reason for their optimal prices to move in the same direction. Finally, we believe that our innovative modelling approach contributes to the operations research methodology. Formulating the firm's joint pricing and production decision as a quadratic program with equality constraints enables us to obtain a state-independent functional form of the solution which ensures tractability of an otherwise very difficult problem. We believe that a similar problem formulation can facilitate analytical tractability of other challenging problems in the field as well.

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10 Appendix

Proof of Lemma 1: Suppose that $\begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is singular, i.e., there exists $\mathbf{x} \neq \mathbf{0}$ such that $\begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \mathbf{x} = \mathbf{0}$. If we partition $\mathbf{x}$ as $\mathbf{x}^T = \begin{pmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T & \mathbf{x}_3^T \end{pmatrix}$, the last equality can be written as

$$\begin{pmatrix} 2\mathbf{I} & \mathbf{0} & \mathbf{D}_1\mathbf{A}^T \\ \mathbf{0} & 2\mathbf{I} & \mathbf{D}_2 \\ \mathbf{A} & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \iff \begin{aligned} 2\mathbf{x}_1 + \mathbf{D}_1\mathbf{A}^T\mathbf{x}_3 &= \mathbf{0}, \\ 2\mathbf{x}_2 + \mathbf{D}_2\mathbf{x}_3 &= \mathbf{0}, \\ \mathbf{A}\mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{0}, \end{aligned} \tag{11}$$

and, therefore,

$$\mathbf{D}_1\mathbf{A}^T\mathbf{D}_2^{-1}\mathbf{A}\mathbf{x}_1 = -\mathbf{x}_1. \tag{12}$$

Note that if $\mathbf{x}_1 = \mathbf{0}$, then (11) implies $\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq \mathbf{0}$, we must have $\mathbf{x}_1 \neq \mathbf{0}$. It then follows from (12) that $\mathbf{x}_1$ is an eigenvector of $\mathbf{D}_1\mathbf{A}^T\mathbf{D}_2^{-1}\mathbf{A}$ and -1 is its eigenvalue. This is a contradiction because $\mathbf{D}_1\mathbf{A}^T\mathbf{D}_2^{-1}\mathbf{A}$ is symmetric and positive semidefinite and thus can have only nonnegative eigenvalues. Hence, our premise that $\begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is singular cannot be true.□

Proof of Proposition 1: The second-stage problem (5) can be written as

$$\mathbf{y}^* = \arg\max_{\mathbf{y}} \left[\mathbf{y}^T\mathbf{D}^{-1}\boldsymbol{\xi} - \mathbf{y}^T\mathbf{D}^{-1}\mathbf{y}\right] \text{ subject to } \mathbf{B}\mathbf{y} = \mathbf{x}^*.$$

Because the rows of $\mathbf{B}$ are linearly independent, this program has always a feasible solution. Further, since $\mathbf{D}^{-1}$ is positive definite, the objective is strictly concave and there exists a unique solution $\mathbf{y}^*$ characterized by the necessary and sufficient Karush-Kuhn-Tucker conditions

$$\begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ -\mathbf{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{x}^* \end{pmatrix}, \tag{13}$$

where $\mathbf{u}$ is the Lagrange multiplier. The optimality condition (13) implies that

$$\mathbb{V}\begin{pmatrix} \mathbf{y}^* \\ -\mathbf{u}^* \end{pmatrix} = \begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} 2\mathbf{I} & \mathbf{B}^T \\ \mathbf{B}\mathbf{D}^T & \mathbf{0} \end{pmatrix}^{-1}, \tag{14}$$

which gives $\mathbb{V}\mathbf{y}^* = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T$.

The first-stage decision problem (4) can be written as

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \mathbb{E}\left[-\mathbf{x}^T\mathbf{c} + \mathbf{y}^{*T}\mathbf{D}^{-1}\left(\boldsymbol{\xi} - \mathbf{y}^*\right)\right]. \tag{15}$$

Since the realized profit is concave in $\mathbf{y}^*$, and each element of $\mathbf{y}^*$ is linear in $\mathbf{x}$, the realized profit is also concave in $\mathbf{x}$. Hence, the expected profit is also concave in $\mathbf{x}$, and the optimal input vector $\mathbf{x}^*$ is characterized by the necessary and sufficient condition $\nabla_{\mathbf{x}}\mathbb{E}\pi\left(\mathbf{x}, \mathbf{y}^*\right) = \mathbf{0}$. The gradient of the expected profit is

$$\begin{aligned} \nabla_{\mathbf{x}}\mathbb{E}\pi\left(\mathbf{x}, \mathbf{y}^*\right) &= -\mathbf{c} + \mathbb{E}\left[\mathbf{J}^T\left(\mathbf{y}^*, \mathbf{x}\right)\mathbf{D}^{-1}\left(\boldsymbol{\xi} - 2\mathbf{y}^*\right)\right], \qquad (16) \\ \text{where } \mathbf{J}\left(\mathbf{y}^*, \mathbf{x}\right) &= \begin{pmatrix} \frac{\partial y_1^*}{\partial x_1} & \cdots & \frac{\partial y_1^*}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial y_{n+m}^*}{\partial x_1} & \cdots & \frac{\partial y_{n+m}^*}{\partial x_m} \end{pmatrix}. \end{aligned}$$

The characterization of the Jacobian $\mathbf{J}\left(\mathbf{y}^*, \mathbf{x}\right)$ can be obtained by implicit differentiation of (13) with respect to $\mathbf{x}$, i.e.,

$$\begin{pmatrix} 2\mathbf{I} & \mathbf{D}\mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{J}\left(\mathbf{y}^*, \mathbf{x}\right) \\ -\mathbf{J}\left(\mathbf{u}^*, \mathbf{x}\right) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}. \tag{17}$$

Notice from (17) that $\mathbf{J}\left(\mathbf{y}^*, \mathbf{x}\right)$ does not depend on $\boldsymbol{\xi}$, and thus can be taken out of the expectation in (16). The necessary and sufficient condition for the optimal $\mathbf{x}^*$, $\nabla_{\mathbf{x}}\mathbb{E}\pi\left(\mathbf{x}, \mathbf{y}^*\right) = \mathbf{0}$, thus simplifies into

$$\mathbf{0} = -\mathbf{c} + \mathbf{J}^T\left(\mathbf{y}^*, \mathbf{x}\right)\mathbf{D}^{-1}\left(\boldsymbol{\mu} - 2\mathbb{E}\mathbf{y}^*\right), \tag{18}$$

where $\mathbf{J}(\mathbf{y}^*, \mathbf{x})$ is implied by (17), and $\mathbb{E}\mathbf{y}^*$ is the solution to (13) with $\boldsymbol{\xi}$ replaced by $\boldsymbol{\mu}$. It is straightforward to verify that the unique solution to the optimality condition (18) is $\mathbf{x}^* = \mathbf{B}\mathbb{E}\mathbf{y}^*$ and $\mathbb{E}\mathbf{y}^* = \frac{1}{2}\left(\boldsymbol{\mu} - \mathbf{D}\mathbf{B}^T\mathbf{c}\right)$. Plugging $\mathbf{x}^* = \mathbf{B}\mathbb{E}\mathbf{y}^*$ and $\mathbb{E}\mathbf{y}^* = \frac{1}{2}\left(\boldsymbol{\mu} - \mathbf{D}\mathbf{B}^T\mathbf{c}\right)$ into (13) and applying some algebra yields $\mathbf{y}^* = \frac{1}{2}\left(\boldsymbol{\mu} - \mathbf{D}\mathbf{B}^T\mathbf{c}\right) + \mathbf{Q}(\boldsymbol{\xi} - \boldsymbol{\mu})$.

Finally, knowing the optimal input and output levels, we can evaluate the firm's expected profit

$$\begin{aligned}\mathbb{E}\pi(\mathbf{x}^*, \mathbf{y}^*) &= \mathbb{E}\left[-\mathbf{c}^T\mathbf{x}^* + \mathbf{y}^{*T}\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*)\right] = \mathbb{E}\left[\mathbf{y}^{*T}\mathbf{D}^{-1}\mathbf{y}^*\right] \\ &= \mathbb{E}\mathbf{y}^{*T}\mathbf{D}^{-1}\mathbb{E}\mathbf{y}^* + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\left(\mathbf{D}^{-1}\right)_{ij}(\mathbb{V}\mathbf{y}^*)_{ij}.\square\end{aligned}$$

Proof of Lemma 2: Problem (6) can be equivalently written as

$$\hat{\mathbf{y}} = \arg\max\left[\mathbf{y}^T\left(\mathbf{D}^{-1}\boldsymbol{\mu} - \mathbf{B}^T\mathbf{c}\right) - \mathbf{y}^T\mathbf{D}^{-1}\mathbf{y}\right] \text{ subject to } \mathbf{y} \geq \mathbf{0}.$$

It is straightforward to show that this is solved by $\hat{\mathbf{y}} = \mathbb{E}\mathbf{y}^*$. The expected profit at optimum can be written as

$$\mathbb{E}\pi(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = -\mathbb{E}\mathbf{y}^{*T}\mathbf{B}^T\mathbf{c} + \mathbb{E}\mathbf{y}^{*T}\mathbb{E}\mathbf{p}(\boldsymbol{\xi}, \mathbb{E}\mathbf{y}^*) = \mathbb{E}\mathbf{y}^{*T}\mathbf{D}^{-1}\mathbb{E}\mathbf{y}^*.\square$$

Proof of Corollary 1: The result follows directly from Proposition 1 and Lemma 2.□

Proof of Lemma 3: The expression for V follows directly from Corollary 1, Proposition 1 and the fact that $\mathbf{D} = \mathbf{I}$.

To derive the expression for $\mathbf{Q}$, we use the fact that the commonality matrix $\mathbf{C} = \mathbf{A}^T\mathbf{A}$, and the technological complementarity matrix

$$\mathbf{Z} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \dots & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \dots & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \dots & \mathbf{1} & \mathbf{0} \end{pmatrix},$$

where all the $\mathbf{0}$ and $\mathbf{1}$ submatrices of $\mathbf{Z}$ are $l \times l$. Our derivation further relies on the following facts:

$$\mathbf{1}\mathbf{A}^T = l^{k-1}\mathbf{1}, \tag{19}$$

$$\mathbf{A}\mathbf{A}^T = l^{k-1}\mathbf{I} + l^{k-2}\mathbf{Z}, \tag{20}$$

$$\mathbf{C}\mathbf{A}^T = l^{k-2}(k-1)\mathbf{1} + l^{k-1}\mathbf{A}^T, \tag{21}$$

$$\mathbf{C}^T\mathbf{C} = \left(k^2 - k\right)l^{k-2}\mathbf{1} + l^{k-1}\mathbf{C}, \tag{22}$$

$$\text{and } \mathbf{1}\mathbf{C} = kl^{k-1}\mathbf{1}. \tag{23}$$

Equalities (19)-(21) are rather straightforward. To prove equality (22), we partition the technology matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & ... & \mathbf{A}_{1l} \\ \mathbf{A}_2 & ... & \mathbf{A}_2 \\ \vdots & & \vdots \\ \mathbf{A}_k & ... & \mathbf{A}_k \end{pmatrix}, \text{ where } \mathbf{A}_{1i}\left[l \times l^{k-1}\right] = \begin{pmatrix} \mathbf{0}\left[(i-1) \times l^{k-1}\right] \\ \mathbf{1}\left[1 \times l^{k-1}\right] \\ \mathbf{0}\left[(l-i) \times l^{k-1}\right] \end{pmatrix}$$

$$\text{and } \mathbf{A}_j\left[l \times l^{k-1}\right] = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & ... & \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & ... & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & ... & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots & \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots & ... & \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} & \vdots & \vdots & \ddots & \ddots & \mathbf{0} & ... & \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} & ... & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

and where all the $\mathbf{0}$ and $\mathbf{1}$ submatrices of $\mathbf{A}_j$ are $1 \times l^{k-j}$. We have

$$\begin{aligned} \mathbf{C} &= \mathbf{A}^T\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11}^T & \mathbf{A}_2^T & ... & \mathbf{A}_k^T \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1l}^T & \mathbf{A}_2^T & ... & \mathbf{A}_k^T \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11} & ... & \mathbf{A}_{1l} \\ \mathbf{A}_2 & ... & \mathbf{A}_2 \\ \vdots & & \vdots \\ \mathbf{A}_k & ... & \mathbf{A}_k \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}_{11}^T\mathbf{A}_{11} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_{11}^T\mathbf{A}_{12} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_{11}^T\mathbf{A}_{1l} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i \\ \mathbf{A}_{12}^T\mathbf{A}_{11} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_{12}^T\mathbf{A}_{12} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_{12}^T\mathbf{A}_{1l} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1l}^T\mathbf{A}_{11} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_{1l}^T\mathbf{A}_{12} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_{1l}^T\mathbf{A}_{1l} + \sum_{i=2}^k \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}_{11}^T\mathbf{A}_{11} & \mathbf{A}_{11}^T\mathbf{A}_{12} & ... & \mathbf{A}_{11}^T\mathbf{A}_{1l} \\ \mathbf{A}_{12}^T\mathbf{A}_{11} & \mathbf{A}_{12}^T\mathbf{A}_{12} & ... & \mathbf{A}_{12}^T\mathbf{A}_{1l} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1l}^T\mathbf{A}_{11} & \mathbf{A}_{1l}^T\mathbf{A}_{12} & ... & \mathbf{A}_{1l}^T\mathbf{A}_{1l} \end{pmatrix} + \sum_{i=2}^k \begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix}. \end{aligned} \tag{24}$$

Since $\mathbf{A}_{1i}^T\mathbf{A}_{1j} = \begin{cases} \mathbf{0} \text{ if } j \neq i \\ \mathbf{1} \text{ if } i = j \end{cases}$, we have

$$\begin{pmatrix} \mathbf{A}_{11}^T\mathbf{A}_{11} & \mathbf{A}_{11}^T\mathbf{A}_{12} & ... & \mathbf{A}_{11}^T\mathbf{A}_{1l} \\ \mathbf{A}_{12}^T\mathbf{A}_{11} & \mathbf{A}_{12}^T\mathbf{A}_{12} & ... & \mathbf{A}_{12}^T\mathbf{A}_{1l} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1l}^T\mathbf{A}_{11} & \mathbf{A}_{1l}^T\mathbf{A}_{12} & ... & \mathbf{A}_{1l}^T\mathbf{A}_{1l} \end{pmatrix} \left[l^k \times l^k\right] = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{25}$$

where the $\mathbf{1}$ and $\mathbf{0}$ submatrices of the last matrix are $l^{k-1} \times l^{k-1}$. Since

$$\mathbf{A}_i^T\mathbf{A}_i = \begin{pmatrix} \mathbf{M}_i & ... & \mathbf{M}_i \\ \vdots & & \vdots \\ \mathbf{M}_i & ... & \mathbf{M}_i \end{pmatrix}, \text{ where } \mathbf{M}_i \left[l^{k-i+1} \times l^{k-i+1}\right] = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

with the $\mathbf{1}$ and $\mathbf{0}$ submatrices of $\mathbf{M}_i$ being $l^{k-i} \times l^{k-i}$, we have

$$\begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix} \left[l^k \times l^k\right] = \begin{pmatrix} \mathbf{M}_i & ... & \mathbf{M}_i \\ \vdots & & \vdots \\ \mathbf{M}_i & ... & \mathbf{M}_i \end{pmatrix}. \tag{26}$$

Combining (24), (25) and (26), we obtain

$$\mathbf{C} = \sum_{i=1}^{k} \begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix},$$

and, therefore,

$$\mathbf{C}^T\mathbf{C} = \left[\sum_{i=1}^{k} \begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix}\right]^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} \left[\begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix} \begin{pmatrix} \mathbf{A}_j^T\mathbf{A}_j & ... & \mathbf{A}_j^T\mathbf{A}_j \\ \vdots & & \vdots \\ \mathbf{A}_j^T\mathbf{A}_j & ... & \mathbf{A}_j^T\mathbf{A}_j \end{pmatrix}\right].$$

Since

$$\begin{pmatrix} \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \\ \vdots & & \vdots \\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i \end{pmatrix} \begin{pmatrix} \mathbf{A}_j^T\mathbf{A}_j & ... & \mathbf{A}_j^T\mathbf{A}_j \\ \vdots & & \vdots \\ \mathbf{A}_j^T\mathbf{A}_j & ... & \mathbf{A}_j^T\mathbf{A}_j \end{pmatrix} = \begin{cases} l^{k-2}\mathbf{1} & \text{if } i \neq j, \\ l^{k-1} \begin{pmatrix} \mathbf{M}_i & ... & \mathbf{M}_i \\ \vdots & & \vdots \\ \mathbf{M}_i & ... & \mathbf{M}_i \end{pmatrix} & \text{if } i = j \end{cases},$$

we have

$$\mathbf{C}^T\mathbf{C} = \left(k^2-k\right) l^{k-2}\mathbf{1}+\sum_{i=1}^{k} l^{k-1}\begin{pmatrix}\mathbf{M}_i & ... & \mathbf{M}_i\\ \vdots & & \vdots\\ \mathbf{M}_i & ... & \mathbf{M}_i\end{pmatrix} = \left(k^2-k\right) l^{k-2}\mathbf{1}+l^{k-1}\sum_{i=1}^{k}\begin{pmatrix}\mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i\\ \vdots & & \vdots\\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i\end{pmatrix},$$

which simplifies into (22).

Equality (23) can be proved as follows:

$$\mathbf{1C} = \sum_{i=1}^{n}\left(\mathbf{C}\right)_{i1}\mathbf{1} = k\sum_{i=1}^{n}\begin{pmatrix}\mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i\\ \vdots & & \vdots\\ \mathbf{A}_i^T\mathbf{A}_i & ... & \mathbf{A}_i^T\mathbf{A}_i\end{pmatrix}_{i1}\mathbf{1} = kl^{k-1}\mathbf{1}.$$

Having established equalities (19)-(23), we are ready to derive matrix $\mathbf{Q}\left[(n+m)\times(n+m)\right]$ defined by

$$\begin{pmatrix}\mathbf{Q} & \mathbf{R}\\ \mathbf{S} & \mathbf{T}\end{pmatrix} = \begin{pmatrix}2\mathbf{I} & \mathbf{B}^T\\ \mathbf{B} & \mathbf{0}\end{pmatrix}^{-1}. \tag{27}$$

If follows from (27) that $\mathbf{Q}$ is the unique solution to

$$\begin{aligned} 2\mathbf{Q}+\mathbf{RB} &= \mathbf{I}\\ \text{and } \mathbf{QB}^T &= \mathbf{0}. \end{aligned} \tag{28}$$

Partitioning $\mathbf{Q}$ into $\mathbf{Q}_1\left[n\times n\right], \mathbf{Q}_2\left[n\times m\right], \mathbf{Q}_3\left[m\times n\right]$ and $\mathbf{Q}_4\left[m\times m\right]$, and $\mathbf{R}$ into $\mathbf{R}_1\left[n\times m\right]$ and $\mathbf{R}_2\left[m\times m\right]$ enables rewriting system (28) as

$$\begin{aligned} 2\begin{pmatrix}\mathbf{Q}_1 & \mathbf{Q}_2\\ \mathbf{Q}_3 & \mathbf{Q}_4\end{pmatrix}+\begin{pmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{I}\end{pmatrix} &= \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\\ \text{and } \begin{pmatrix}\mathbf{Q}_1 & \mathbf{Q}_2\\ \mathbf{Q}_3 & \mathbf{Q}_4\end{pmatrix}\begin{pmatrix}\mathbf{A}^T\\ \mathbf{I}\end{pmatrix} &= \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \end{aligned}$$

which yields

$$\mathbf{Q}_1 = \frac{1}{2}\left(\mathbf{I}+\mathbf{C}\right)^{-1},\ \mathbf{Q}_2 = -\mathbf{Q}_1\mathbf{A}^T,\ \mathbf{Q}_3 = \mathbf{Q}_2^T \text{ and } \mathbf{Q}_4 = -\mathbf{Q}_3\mathbf{A}^T. \tag{29}$$

The expression for $\mathbf{Q}$ follows from (29) and equalities (19)-(23).

The remaining proofs rely on manipulating columns of matrix $\mathbf{Q}$ so it is useful to write down their generic form. Let $u\in\{1,...,n\}$ be an end-product index and let $v\in\{1,...,m\}$ be a component index. The u^{th} and $(n+v)^{th}$ columns of $\mathbf{Q}$ are, respectively,

$$\mathbf{q}_u = \begin{pmatrix} k\alpha - \beta\left(\mathbf{C}\right)_{1u} \\ k\alpha - \beta\left(\mathbf{C}\right)_{2u} \\ \vdots \\ k\alpha - \beta\left(\mathbf{C}\right)_{(u-1)u} \\ k\alpha - \beta\left(\mathbf{C}\right)_{uu} + \frac{1}{2} \\ k\alpha - \beta\left(\mathbf{C}\right)_{(u+1)u} \\ \vdots \\ k\alpha - \beta\left(\mathbf{C}\right)_{nu} \\ \alpha - \beta\left(\mathbf{A}\right)_{1u} \\ \alpha - \beta\left(\mathbf{A}\right)_{2u} \\ \vdots \\ \alpha - \beta\left(\mathbf{A}\right)_{mu} \end{pmatrix} \tag{30}$$

$$\text{and } \mathbf{q}_{n+v} = \begin{pmatrix} \alpha - \beta\left(\mathbf{A}^T\right)_{1v} \\ \alpha - \beta\left(\mathbf{A}^T\right)_{2v} \\ \vdots \\ \alpha - \beta\left(\mathbf{A}^T\right)_{nv} \\ \left.\begin{matrix} l^{k-2}\beta - l^{k-1}\alpha \\ \vdots \\ l^{k-2}\beta - l^{k-1}\alpha \end{matrix}\right\} l(\gamma_v - 1) \\ \left.\begin{matrix} -l^{k-1}\alpha \\ \vdots \\ -l^{k-1}\alpha \end{matrix}\right\} l \\ \left.\begin{matrix} l^{k-2}\beta - l^{k-1}\alpha \\ \vdots \\ l^{k-2}\beta - l^{k-1}\alpha \end{matrix}\right\} l(k - \gamma_v) \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ l^{k-1}\beta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow (n+v)^{th} \text{ row}, \tag{31}$$

where $\alpha = \frac{(k-1)l^{k-2}}{2\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}$, $\beta = \frac{1}{2\left(1+l^{k-1}\right)}$ and γ_v is the index of the component class of component v.$\square$

Proof of Proposition 2: We know from Lemma 3 that $V=\sum_{i=1}^{n+m}\left(\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T\right)_{ii}$. Therefore,

$$\frac{\partial V}{\partial \rho_{uv}}=\frac{\partial}{\partial \rho_{uv}}\sum_{i=1}^{n+m}\left(\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T\right)_{ii}=\sum_{i=1}^{n+m}\left(\mathbf{Q}\frac{\partial \mathbf{\Sigma}}{\partial \rho_{uv}}\mathbf{Q}^T\right)_{ii}, \text{ where } \left(\frac{\partial \mathbf{\Sigma}}{\partial \rho_{uv}}\right)_{ij}=\begin{cases}\sigma_u\sigma_v \text{ if } ij=uv \text{ or } vu\\ 0 \text{ otherwise}\end{cases}$$

Hence

$$\frac{\partial V}{\partial \rho_{uv}}=2\sigma_u\sigma_v\sum_{i=1}^{n+m}\left[(\mathbf{Q})_{iu}(\mathbf{Q})_{iv}\right]=2\sigma_u\sigma_v\mathbf{q}_u^T\mathbf{q}_v.$$

Using (30), we have

$$\mathbf{q}_u^T\mathbf{q}_v=\sum_{i=1}^{n}\left[\left(k\alpha-\beta(\mathbf{C})_{iu}\right)\left(k\alpha-\beta(\mathbf{C})_{iv}\right)\right]+\sum_{i=1}^{m}\left[\left(\alpha-\beta(\mathbf{A})_{iu}\right)\left(\alpha-\beta(\mathbf{A})_{iv}\right)\right]+k\alpha-\beta(\mathbf{C})_{uv}.$$

Using (22) and (23), this simplifies into

$$\mathbf{q}_u^T\mathbf{q}_v=\frac{\left(k^2-k\right)l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}-\frac{1}{4\left(1+l^{k-1}\right)}(\mathbf{C})_{uv},$$

and the result follows.□

Proof of Proposition 3: We know from the proof of Proposition 2 that

$$\frac{\partial V}{\partial \rho_{u,n+v}}=2\sigma_u\sigma_{n+v}\mathbf{q}_u^T\mathbf{q}_{n+v}.$$

Using (30) and (31), we have

$$\begin{aligned}\mathbf{q}_u^T\mathbf{q}_{n+v} \;=\;& \sum_{i=1}^{n}\left[k\alpha-\beta(\mathbf{C})_{iu}\right]\left[\alpha-\beta\left(\mathbf{A}^T\right)_{iv}\right]-l^{k-1}\alpha\sum_{i=1}^{kl}\left(\alpha-\beta(\mathbf{A})_{iu}\right)+l^{k-2}\beta\sum_{i=1}^{l(\gamma_v-1)}\left[\alpha-\beta(\mathbf{A})_{iu}\right]\\ &+l^{k-2}\beta\sum_{i=l\gamma_v+1}^{lk}\left[\alpha-\beta(\mathbf{A})_{iu}\right]+\left(\frac{1}{2}+l^{k-1}\beta\right)\left(\alpha-\beta(\mathbf{A})_{vu}\right).\end{aligned}$$

Using (21)-(23), this simplifies into

$$\mathbf{q}_u^T\mathbf{q}_{n+v}=\frac{1}{2}\alpha-\frac{1}{2}\beta(\mathbf{A})_{vu},$$

and the result follows.□

Proof of Proposition 4: It follows from the proof of Proposition 2 that

$$\frac{\partial V}{\partial \rho_{n+u,n+v}}=2\sigma_{n+u}\sigma_{n+v}\mathbf{q}_{n+u}^T\mathbf{q}_{n+v}.$$

To evaluate $\mathbf{q}_{n+u}^T\mathbf{q}_{n+v}$, we use (31) distinguishing two cases:

(i) If $(\mathbf{Z})_{uv}=0$, we have

$$\begin{aligned}\mathbf{q}_{n+u}^T\mathbf{q}_{n+v} &= \sum_{i=1}^{n}\left[\left(\alpha-\beta\left(\mathbf{A}^T\right)_{iu}\right)\left(\alpha-\beta\left(\mathbf{A}^T\right)_{iv}\right)\right]+\alpha^2kl^{2k-1}-2\alpha\beta kl^{2k-2}+\beta^2\left(k-1\right)l^{2k-3}\\ &= \frac{(1-k)\,l^{2k-3}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}<0.\end{aligned}$$

(ii) If $(\mathbf{Z})_{uv}=1$, we have

$$\begin{aligned}\mathbf{q}_{n+u}^T\mathbf{q}_{n+v} &= \sum_{i=1}^{n}\left[\left(\alpha-\beta\left(\mathbf{A}^T\right)_{iu}\right)\left(\alpha-\beta\left(\mathbf{A}^T\right)_{iv}\right)\right]+\alpha^2kl^{2k-1}-2\alpha\beta kl^{2k-2}+\beta^2kl^{2k-3}\\ &= \frac{l^{2k-3}+l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}>0,\end{aligned}$$

and the result follows.□

Proof of Proposition 5: We know from Lemma 3 that $V=\sum_{i=1}^{n+m}\left(\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T\right)_{ii}$. Therefore,

$$\frac{\partial V}{\partial\sigma_u} = \frac{\partial}{\partial\sigma_u}\sum_{i=1}^{n+m}\left(\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T\right)_{ii}=\sum_{i=1}^{n+m}\left(\mathbf{Q}\frac{\partial\mathbf{\Sigma}}{\partial\sigma_u}\mathbf{Q}^T\right)_{ii},$$

$$\text{where } \frac{\partial\mathbf{\Sigma}}{\partial\sigma_u} = \begin{pmatrix} 0 & ... & 0 & \sigma_1\rho_{1u} & 0 & ... & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & ... & 0 & \sigma_{u-1}\rho_{u-1,u} & 0 & ... & 0\\ \sigma_1\rho_{1u} & ... & \sigma_{u-1}\rho_{u-1,u} & 2\sigma_u & \sigma_{u+1}\rho_{u+1,u} & ... & \sigma_{n+m}\rho_{n+m,u}\\ 0 & ... & 0 & \sigma_{u+1}\rho_{u+1,u} & 0 & ... & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & ... & 0 & \sigma_{n+m}\rho_{n+m,u} & 0 & ... & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned}\frac{\partial V}{\partial\sigma_u} &= \sum_{i=1}^{n+m}2\sigma_i\rho_{iu}\mathbf{q}_u^T\mathbf{q}_i\\ &= \sum_{\substack{i=1\\ i\neq u}}^{n}2\sigma_i\rho_{iu}\mathbf{q}_u^T\mathbf{q}_i+2\sigma_u\mathbf{q}_u^T\mathbf{q}_u+\sum_{i=n+1}^{n+m}2\sigma_i\rho_{iu}\mathbf{q}_u^T\mathbf{q}_i.\end{aligned}$$

Using (30) and (31), we have

$$\begin{aligned}\frac{\partial V}{\partial\sigma_u} &= \sum_{\substack{i=1\\ i\neq u}}^{n}2\sigma_i\rho_{iu}\left(\frac{\left(k^2-k\right)l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}-\frac{1}{4\left(1+l^{k-1}\right)}\left(\mathbf{C}\right)_{ui}\right)\\ &\quad+2\sigma_u\left(\frac{\left(k^2-k\right)l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}-\frac{k}{4\left(1+l^{k-1}\right)}+\frac{1}{4}\right)\\ &\quad+\sum_{i=n+1}^{n+m}2\sigma_i\rho_{iu}\left(\frac{(k-1)\,l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)}-\frac{1}{4\left(1+l^{k-1}\right)}\left(\mathbf{A}\right)_{i-n,u}\right),\end{aligned}$$

and the result follows.□

Proof of Proposition 6: We know from the proof of Proposition 5 that

$$\begin{aligned}\frac{\partial V}{\partial \sigma_{n+v}} &= \sum_{i=1}^{n+m} 2\sigma_i \rho_{i,n+v} \mathbf{q}_{n+v}^T \mathbf{q}_i \\ &= \sum_{i=1}^{n} 2\sigma_i \rho_{i,n+v} \mathbf{q}_{n+v}^T \mathbf{q}_i + \sum_{\substack{i=n+1 \\ (\mathbf{Z})_{i-n,v}=1}}^{n+m} 2\sigma_i \rho_{i,n+v} \mathbf{q}_{n+v}^T \mathbf{q}_i \\ &\quad + \sum_{\substack{i=n+1 \\ (\mathbf{Z})_{i-n,v}=0 \\ i-n\neq v}}^{n+m} 2\sigma_i \rho_{i,n+v} \mathbf{q}_{n+v}^T \mathbf{q}_i + 2\sigma_{n+v} \mathbf{q}_{n+v}^T \mathbf{q}_{n+v}.\end{aligned}$$

Using (30) and (31), we have

$$\begin{aligned}\frac{\partial V}{\partial \sigma_{n+v}} &= \sum_{i=1}^{n} 2\sigma_i \rho_{i,n+v} \left(\frac{(k-1)\, l^{k-2}}{4\,(1+l^{k-1})\,(1+kl^{k-1})} - \frac{1}{4\,(1+l^{k-1})} (\mathbf{A})_{vi} \right) \\ &\quad + \sum_{\substack{i=n+1 \\ (\mathbf{Z})_{i-n,v}=1}}^{n+m} 2\sigma_i \rho_{i,n+v} \frac{l^{2k-3} + l^{k-2}}{4\,(1+l^{k-1})\,(1+kl^{k-1})} \\ &\quad + \sum_{\substack{i=n+1 \\ (\mathbf{Z})_{i-n,v}=0 \\ i-n\neq v}}^{n+m} 2\sigma_i \rho_{i,n+v} \frac{(1-k)\, l^{2k-3}}{4\,(1+l^{k-1})\,(1+kl^{k-1})} \\ &\quad + 2\sigma_{n+v} \frac{l^{k-1} + l^{2k-3} + kl^{2k-2} - kl^{2k-3}}{4\,(1+l^{k-1})\,(1+kl^{k-1})},\end{aligned}$$

and the result follows.□

Proof of Proposition 7: The price vector $\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*) = \boldsymbol{\xi} - \mathbf{y}^*$ can be rewritten using (13) as

$$\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*) = \mathbf{y}^* - \mathbf{B}^T \mathbf{u}^* = \begin{pmatrix} \mathbf{I} & \mathbf{B}^T \end{pmatrix} \begin{pmatrix} \mathbf{y}^* \\ -\mathbf{u}^* \end{pmatrix}.$$

Hence, its covariance matrix is $\mathbb{V}\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*) = \begin{pmatrix} \mathbf{I} \; \mathbf{B}^T \end{pmatrix} \mathbb{V} \begin{pmatrix} \mathbf{y}^* \\ -\mathbf{u}^* \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{B} \end{pmatrix}$. Using (14), this can be written as

$$\mathbb{V}\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*) = \begin{pmatrix} \mathbf{I} & \mathbf{B}^T \end{pmatrix} \begin{pmatrix} 2\mathbf{I} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} 2\mathbf{I} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} \\ \mathbf{B} \end{pmatrix}.$$

Using (27) and the fact that $\mathbf{S} = \mathbf{R}^T$, this can be rewritten as

$$\mathbb{V}\mathbf{p}(\boldsymbol{\xi}, \mathbf{y}^*) = \left(\mathbf{Q} + \mathbf{B}^T \mathbf{R}^T \right) \boldsymbol{\Sigma} \left(\mathbf{Q} + \mathbf{R}\mathbf{B} \right).$$

Using (28), this simplifies into

$$\mathbb{V}\mathbf{p}\left(\boldsymbol{\xi},\mathbf{y}^*\right)=\left(\mathbf{I}-\mathbf{Q}\right)\boldsymbol{\Sigma}\left(\mathbf{I}-\mathbf{Q}\right).$$

When $\boldsymbol{\Sigma}=\mathbf{I}$, this further simplifies into $\mathbb{V}\mathbf{p}\left(\boldsymbol{\xi},\mathbf{y}^*\right)=\mathbf{I}-2\mathbf{Q}+\mathbf{Q}\mathbf{Q}$. Therefore, if $u\neq v$ and $u,v\leq n$, we have

$$\left(\mathbb{V}\mathbf{p}\left(\boldsymbol{\xi},\mathbf{y}^*\right)\right)_{uv}=-2\left(\mathbf{Q}\right)_{uv}+\mathbf{q}_u^T\mathbf{q}_v=\frac{3}{4\left(1+l^{k-1}\right)}\left(\mathbf{C}\right)_{uv}-\frac{3\left(k^2-k\right)l^{k-2}}{4\left(1+l^{k-1}\right)\left(1+kl^{k-1}\right)},$$

and the result follows.□

Proof of Lemma 4: Proposition 1 implies that the optimal vector of end-product outputs can be written as $\mathbf{y}_1^*=\frac{1}{2}\left(\boldsymbol{\mu}_1-\mathbf{D}_1\mathbf{A}^T\mathbf{c}\right)+\left(\mathbf{Q}_1\ \mathbf{Q}_2\right)\left(\boldsymbol{\xi}-\boldsymbol{\mu}\right)$, and condition (9) follows.□