An assemble-to-order system with component substitution

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Abstract

In assemble-to-order systems, it is often possible to substitute high-end components for low-end components when the latter are out of stock. For example, when faced with a stockout of European-specific power supplies for network laser printers, Hewlett-Packard might use a more expensive universal power supply in its place. How to manage inventories of components in such a system is of managerial interest. We model this scenario for a simple system of two end products, each composed of two components, one of which can be downward substituted. We demonstrate concavity of the expected profit function and find the first order conditions for jointly optimal inventory levels for all the components. We prove several properties of the system, and we develop bounds on the optimal order quantities. The analysis facilitates insights into the management of such a system.

1 Introduction

In many manufacturing operations, an expensive component can be substituted for a less expensive component if the latter is out of stock. Examples of this type of substitution abound in many industries. Hewlett-Packard, for instance, used distinct power supplies for the North American and European printer markets. The difficulty of predicting demand of network laser printers, coupled with fourteen week leadtimes for power supplies, led to high inventories and high stockouts (Lee, 1997). A universal power supply could replace both dedicated power supplies, but its high cost motivated a close look at the potential inventory benefits. HP's decision was fascinating. They decided to offer a dedicated North American version and a universal version – the former clearly for the North American market, and the latter for the European market and for excess North American demand. Therefore, they designed only two versions, and yet obtained significant risk pooling benefits. Moreover, they relied heavily on the universal version early in the product life cycle, when uncertainty about demand was very high, and late in the life cycle, when the cost of excess inventory was high. In between, during the mature phase of the life cycle, demand was high and risk pooling benefits were low, so they relied primarily on the less expensive North American version for that market (Lee and Sasser, 1995). Many other firms employ substitution policies, although perhaps less elaborate ones. Now, once these strategic design decisions are made, managers must determine inventory levels for both the substitutable component and the substituted component. This paper examines optimal inventory decisions in the presence of component substitution.

The situation we model is called assemble-to-order (ATO). ATO is a hybrid strategy where parts and subassemblies are purchased, or produced at relatively long lead times, held in inventory, and then assembled into finished goods as customer demand for specific combinations arrives. Similar strategies have been termed "delayed customization" and "postponement" (Feitzinger and Lee (1997), Garg and Tang (1997), Aviv and Federgruen (2001a, b), Brown, Lee, and Petrakian (2000), Johnson and Anderson (2000), van Hoek (2001)). Therefore, the manager must take two decisions: how many components to purchase before final demand is known, and once demand materializes, how to allocate those components to final demand in a profit maximizing manner. The focus of this paper is on the inventory procurement decision.

There is an extensive literature on the end-product substitution problem, a review of which can be found in Hale (2003). The origins of the analysis of substitution can be traced to the assortment problem. In the assortment problem, a firm has the ability to produce n different items. However, it must satisfy all demand while producing only m items, where $m < n$. In this problem any superior (and more costly) item may be substituted to satisfy demand for an inferior item with the objective of minimizing waste and therefore total cost. This could be termed *firm-driven substitution*. The assortment problem was first solved by Sadowski (1960) using dynamic programming in the context of the optimal selection of steel beam sizes. Rutenberg (1971) introduced two-way substitutability for the deterministic demand case. Pentico (1974) extended the problem to the stochastic demand case and introduced priority substitution schemes to simplify the problem. McGillivray and Silver (1978) were the first to discuss the effect of finished product substitution on inventory control rules. They assumed all products were of equal value (but different capability). Parlar and Goyal (1984) investigated two-way substitution for two end products in a generalized version of the single period newsboy problem. Pasternack and Drezner (1991) included salvage values and shortage costs as they addressed the case in which revenue incurred by the sale of a substituted good may be less than that of a normal good. Cattani (1998) modeled a similar system where products can be substituted without cost, and developed conditions for which the expected profit function was concave with respect to product inventory levels. Rutten (1995) addressed a related problem in the context of the food industry. He examined raw material substitution and employed linear programming to optimize recipes. Bassok, Anupindi, and Akella (1999) studied the single-period and infinite horizon, multiproduct, downward substitution problem and proved concavity of the profit function. Gurnani and Drezner (2000) considered a deterministic nested substitution problem with costs for substituting one product for another.

Mahajan and van Ryzin (2001) addressed substitution in the context of determining stocking assortments at the retail store. This is *customer-driven substitution* in that customers who do not find the product they want may substitute another product for it. (See Netessine and Rudi, 2002.) Finally, Rajaram and Tang (2001) looked at the impact of product substitution on retail merchandising. In all of these papers (both firm-driven and customer-driven substitution), except Rutten (1995), substitution occurs at the end product level only. We extend this literature by focusing on component inventory levels and by allowing for differing component costs.

The ATO problem is closely related to the component commonality problem because, in theory, the firm could replace the substitutable and substituted components with a single common component. In fact, in a typical discussion of the Hewlett-Packard case (Lee, 1997), students argue the commonality solution. (We might add that in the session wrapup they are startled and impressed by HP's mixed, substitution strategy.) Research on the commonality problem has been extensive, beginning with Collier (1981), Baker (1985), and Baker, Magazine, and Nuttle (1986). Recently, Rudi (1998) solved the single period component commonality problem analytically for two products and up to two common components.

Our research builds on and extends the work of Bassok et al. (1999) and Rudi (1998). By incorporating product-specific components, we extend Bassok et al. (1999); and by incorporating a substitution decision with two components, in place of one common component, we extend Rudi (1998).

Prior to the current research, the complexity of the analysis on this problem has precluded the development of insights and results. We analyze a simple model to gain new results and managerial insights. In this pursuit we sacrifice some realism by restricting the size of the problem. Our hope is that these results will be a useful building block for developing implementable heuristics for the more general case.

The main contributions of this paper are as follows. The profit maximization problem for two products is formulated as a two stage stochastic program which is demonstrated to be jointly concave in the order quantities. Through manipulation of this program, the problem is reformulated into an equivalent one-stage stochastic program. The optimality conditions are given in the form of analytical first order conditions, which also give important managerial insight into the trade-offs involved. Further insights are provided by several analytical properties of the optimal solution and by bounds on the optimal order quantities of the components. The remainder of the paper is organized as follows. In Section 2, we define the problem and present the optimization model. In Section 3, we prove a number of properties of the optimal solution, as well as the first order conditions and bounds. Finally, in Section 4, we summarize our results and discuss future research.

2 Problem Definition

The system we model is composed of two end products, each of which is made up of two components. Let the products be indexed by $i = 1, 2$ where 1 is the universal version, or more generally the "luxury version," and 2 is the dedicated or "economy" version. One of the components for product 1 performs similar functions as its counterpart for product 2. Therefore, if there is a stockout, the more expensive component can be substituted for the cheaper component. We shall call these components *substitutable* component 1 and *substituted* component 2, or, together, "sub components," and shall denote them by lower case letters. The other two components, which we shall call *specific* components, will be denoted by upper case letters. These are specific to the products and cannot be substituted. The problem we address is how much of each of the four components to purchase, given that demands for the end products are not known until after the purchase decisions are made. Demands for each end product follow a known continuous probability distribution. We assume that unmet demands for end products are lost.

The sequence of events is as follows. First, components are acquired in quantities Q_i and q_i, at unit costs of C_i and c_i, respectively. After observing demands, D_i, components are allocated to assembly to maximize profit. Each sale generates revenue R_i, and any remaining components are salvaged at unit salvage value S_i and s_i, respectively. To summarize the notation:

Q_i = Order quantity for the specific component for product i (decision variable)

q_i = Order quantity for the “sub component” for product i (decision variable), i.e., the substitutable and substituted components

C_i = Unit cost for specific component i

c_i = Unit cost for sub component i

S_i = Salvage value for specific component i

s_i = Salvage value for sub component i

R_i = Unit revenue for product i

D_i = Demand for product i (random variable)

Notation in boldface indicates the corresponding vector.

The bill of material is illustrated by the following figure:

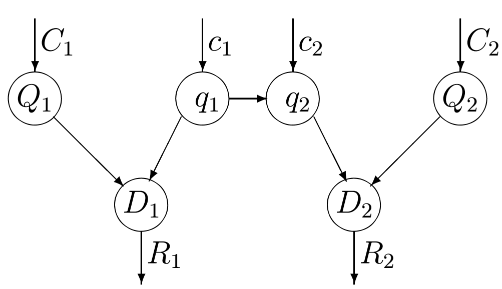

Figure 1: Illustration of bill of material with corresponding parameters

We assume that $R_1 > R_2$, and $s_1 > s_2$, both of which follow directly from the definition of the luxury and economy products and their components. A similar assumption, $R_1 - R_2 > S_1 - S_2$, states that the difference in revenues between the products is greater than the difference in the salvage values of the product-specific components. We highlight this assumption in particular because it facilitates the proof of Property 2. Likewise, we assume that $c_1 - c_2 > s_1 - s_2$, which is generally true since the salvage value is usually a fixed percentage of the unit cost in the ATO environment. To avoid trivialities, we assume that $R_1 > S_1 + s_1$ and $R_2 > S_2 + s_1$, i.e., it is not beneficial to buy components simply to sell them for salvage, even if the more expensive component is substituted. Finally, to facilitate the proof of Property 5, we assume that $R_1 - S_1 - s_1 > c_2 - s_2$. If this assumption is false, the upper bound on Q_1^* will not necessarily hold.

The optimization problem, then, is the following.

$$\max_{\mathbf{Q},\mathbf{q}} \pi = \max_{\mathbf{Q},\mathbf{q}} \left\{ -C_1 Q_1 - c_1 q_1 - C_2 Q_2 - c_2 q_2 + E\bar{K} \right\} \tag{1}$$

where

$$\begin{aligned} \bar{K} = \max_{\mathbf{v}} \quad & R_1 v_{11} + R_2(v_{12} + v_{22}) + S_1(Q_1 - v_{11}) + s_1(q_1 - v_{11} - v_{12}) \\ & + S_2(Q_2 - v_{12} - v_{22}) + s_2(q_2 - v_{22}) \end{aligned} \tag{2}$$

subject to

$$v_{11} \le D_1 \tag{3}$$

$$v_{12} + v_{22} \le D_2 \tag{4}$$

$$v_{11} \le Q_1 \tag{5}$$

$$v_{11} + v_{12} \le q_1 \tag{6}$$

$$v_{12} + v_{22} \le Q_2 \tag{7}$$

$$v_{22} \le q_2 \tag{8}$$

where v_{ij} represents sales of product j, as composed of specific component j and sub component i. So, v_{11} is the sales of product 1, assembled from specific component 1 and sub component 1; v_{22} is the sales of product 2, assembled from specific component 2 and sub component 2; and v_{12} is the sales of product 2, assembled from specific component 2 and sub component 1.

The objective function, (1), captures the cost of buying the components and the expected revenues from sales and salvage. (2) maximizes sales and salvage values; and the constraints, (3) - (8), insure that sales do not exceed demand or component inventories. Stockout penalties can also be incorporated into this analysis, but have been omitted for simplicity of exposition.

3 Analysis

The optimization problem can be rewritten as:

$$\max_{\mathbf{Q},\mathbf{q}} \pi = \max_{\mathbf{Q},\mathbf{q}} \{-(C_1 - S_1)Q_1 - (c_1 - s_1)q_1 - (C_2 - S_2)Q_2 - (c_2 - s_2)q_2 + EK\} \quad (9)$$

where

$$K = \max_{\mathbf{v}} \quad (R_1 - S_1 - s_1)v_{11} + (R_2 - s_1 - S_2)v_{12} + (R_2 - S_2 - s_2)v_{22} \quad (10)$$

subject to

$$v_{11} \leq D_1 \quad (11)$$

$$v_{12} + v_{22} \leq D_2 \quad (12)$$

$$v_{11} \leq Q_1 \quad (13)$$

$$v_{11} + v_{12} \leq q_1 \quad (14)$$

$$v_{12} + v_{22} \leq Q_2 \quad (15)$$

$$v_{22} \leq q_2 \quad (16)$$

To find optimal values for the four decision variables, it is useful to identify several properties of the optimal solutions. These properties also lend insight into the behavior of the system, and they are useful in finding the first order conditions. Let superscript $*$ denote the optimal value of the respective decision variables. We have:

Property 1 *The optimal solution satisfies the following properties:*

(a) $Q_1^* \leq q_1^*$.

(b) $Q_2^* \geq q_2^*$.

Proof Property 1 follows from the two-stage formulation (9). In particular, for (a) note (13) and (14), and for (b) note (15) and (16).□

The only component that has several alternative uses is substitutable component 1. From the above formulation of the decision problem, the greedy allocation of q_1 given in the following property is optimal:

Property 2 *The optimal solution satisfies the following properties:*

(a) *If q_1 is a scarce resource, prioritize product 1.*

(b) *If the demand for product 2 is satisfied, prioritize to salvage substitutable component 1.*

Proof The first follows from the assumption that $R_1 - R_2 > S_1 - S_2$, or equivalently, $R_1 - S_1 - s_1 > R_2 - S_2 - s_1$ (with (14) being binding). The second follows from $R_2 - S_2 - s_1 > R_2 - S_2 - s_2$ (i.e. $s_1 > s_2$).□

In other words, sell as much of the luxury product as possible, and then sell as much of the economy product as possible, prioritizing sub component 2 over any remaining sub components 1. Finally, salvage any remaining components.

For optimization purposes, we give the following proposition:

Proposition 1 *π is jointly concave in the order quantities.*

Proof From the theory of linear programming, the value of a linear maximization program is concave in the right hand sides of the constraints (Hillier and Lieberman, 1986, page 438-439; Bradley, Hax and Magnanti, 1977). Thus, for a given demand realization, K is jointly concave in the order quantities (the right hand sides of the constraints (13) - (16)). The expectation of a concave function is itself concave, so EK is concave. The sum of concave functions is itself concave. It follows that π is jointly concave in the order quantities.□
Proposition 1 ensures that by satisfying the first order conditions, we have a global optimum.

Using Properties 1 and 2, we can rewrite the two-stage program as a one-stage stochastic program:

$$\begin{aligned}\pi &= E\{R_1\min(D_1,Q_1)+R_2\min(D_2,Q_2,q_1+q_2-\min(D_1,Q_1))\\ &\quad +S_1(Q_1-D_1)^+ + s_1(q_1-\min(D_1,Q_1)-(\min(D_2,Q_2)-q_2)^+)^+\\ &\quad +S_2(Q_2-\min(D_2,q_1+q_2-\min(D_1,Q_1)))^+\\ &\quad +s_2(q_2-D_2)^+\} - C_1Q_1 - c_1q_1 - C_2Q_2 - c_2q_2 \end{aligned} \tag{17}$$

Now consider this thought experiment. From Figure 1 and the system description, it would appear obvious that, for product 1, one would want to order more of the substitutable component than the specific component; i.e., $q_1^* > Q_1^*$. Extra units of this substitutable component can then be used in product 2, thereby reducing the need for units of q_2. This is the so-called "risk pooling" effect. Property 3 proves that this is in fact *not* the case.

Property 3 $Q_1^* = q_1^*$.

Proof Let (Q_1, q_1, Q_2, q_2) be an arbitrary solution where $q_1 \geq Q_1$ and Property 1 is satisfied. Let $\epsilon \in (0, q_1 - Q_1]$ and $\pi(\epsilon) \equiv \pi(Q_1, q_1-\epsilon, Q_2, q_2+\epsilon)$. We have

$$\pi(0) < \pi(0) - (s_1 - s_2)E\{\min(\epsilon, (\epsilon + q_2 - D_2)^+)\} + (c_1 - c_2)\epsilon = \pi(\epsilon)$$

Given our assumptions about the parameters, it follows by contradiction that (Q_1, q_1, Q_2, q_2) cannot be an optimal solution, which completes the proof. Also note that $\pi(\epsilon)$ is increasing in ϵ, i.e. the optimal solution has $\epsilon = q_1 - Q_1$.□

The intuition of this result is interesting. The only value derived from additional units of q_1 (that is, the difference $q_1 - Q_1$) is for making product 2. However, if this is true, why not simply buy more units of q_2 instead, thereby saving $c_1 - c_2$ dollars per unit? Clearly, it is more profitable to use the less expensive q_2 units. Therefore, $q_1^* = Q_1^*$.

Using this property, we can simplify the expected profit expression to one with only three decision variables:

$$\begin{aligned}\pi &= E\{R_1 \min(D_1, Q_1) + R_2 \min(D_2, Q_2, q_2 + (Q_1 - D_1)^+) \\ &\quad + S_1(Q_1 - D_1)^+ + s_1((Q_1 - D_1)^+ - (\min(D_2, Q_2) - q_2)^+)^+ \\ &\quad + S_2(Q_2 - \min(D_2, q_2 + (Q_1 - D_1)^+))^+ \\ &\quad + s_2(q_2 - D_2)^+\} - (C_1 + c_1)Q_1 - C_2Q_2 - c_2q_2 \qquad (18)\end{aligned}$$

3.1 First order conditions

By Proposition 1, the first order conditions guarantee global optima. Note that the first order conditions only need to be evaluated for combinations of order quantities that satisfy Properties 1 and 3.

We begin with Q_1. Using the techniques developed in Rudi (1998), and illustrated in Rudi and Pyke (2000), the partial derivative of the profit function with respect to Q_1 is:

$$\frac{\partial \pi}{\partial Q_1} = R_1 \Pr(D_1 > Q_1) + R_2 \Pr(D_1 < Q_1, Q_1 + q_2 - D_1 < \min(D_2, Q_2))$$

$$
\begin{aligned}
&+S_1 \Pr(D_1 < Q_1) + s_1 \Pr(D_1 < Q_1, Q_1 - D_1 > \min(D_2, Q_2) - q_2) \\
&-S_2 \Pr(D_1 < Q_1, Q_1 + q_2 - D_1 < D_2, Q_1 + q_2 - D_1 < Q_2) \\
&-(C_1 + c_1) \qquad (19)
\end{aligned}
$$

Writing the probability expressions long hand yields:

$$
\begin{aligned}
\frac{\partial \pi}{\partial Q_1} \;=\; & R_1(1 - \Pr(D_1 < Q_1)) + R_2[\Pr(D_1 < Q_1, D_2 < Q_2, D_1 + D_2 > Q_1 + q_2) \\
& + \Pr(D_1 < Q_1, D_2 > Q_2, D_1 > Q_1 + q_2 - Q_2)] \\
& + S_1 \Pr(D_1 < Q_1) + s_1[\Pr(D_1 < Q_1, D_2 < Q_2, D_1 + D_2 < Q_1 + q_2) \\
& + \Pr(D_1 < Q_1, D_2 > Q_2, D_1 < Q_1 + q_2 - Q_2)] \\
& - S_2[\Pr(D_1 < Q_1, D_2 < Q_2, D_1 + D_2 > Q_1 + q_2) \\
& + \Pr(D_1 < Q_1, D_2 > Q_2, D_1 > Q_1 + q_2 - Q_2)] - (C_1 + c_1) \qquad (20)
\end{aligned}
$$

By manipulation of $s_1[\cdot]$ and collecting similar terms, we get:

$$
\begin{aligned}
\frac{\partial \pi}{\partial Q_1} \;=\; & R_1 - C_1 - c_1 - (R_1 - S_1 - s_1) \Pr(D_1 < Q_1) \\
& + (R_2 - S_2 - s_1)[\Pr(D_1 < Q_1, D_2 < Q_2, D_1 + D_2 > Q_1 + q_2) \\
& + \Pr(D_1 < Q_1, D_2 > Q_2, D_1 > Q_1 + q_2 - Q_2)] \qquad (21)
\end{aligned}
$$

Equating to zero yields the first order condition:

$$
\begin{aligned}
& \Pr(D_1 < Q_1) - \frac{R_2 - S_2 - s_1}{R_1 - S_1 - s_1}[\Pr(D_1 < Q_1, D_2 < Q_2, D_1 + D_2 > Q_1 + q_2) \\
& + \Pr(D_1 < Q_1, D_2 > Q_2, D_1 > Q_1 + q_2 - Q_2)] = \frac{R_1 - C_1 - c_1}{R_1 - S_1 - s_1} \qquad (22)
\end{aligned}
$$

This complex equation holds interesting insight. If we disregard the second term on the left-hand side, we obtain the optimal simple newsvendor solution for the no-substitution case, i.e., where sub component 1 can not be used to substitute for sub component 2. The underage cost, c_u, is the lost margin, $R_1 - C_1 - c_1$; and the overage cost, c_o, is cost of buying too many components, $C_1 + c_1 - S_1 - s_1$. This quantity is then adjusted *up* by the second term on the left-hand side to account for the benefit of using sub component 1 as a substitute for sub component 2.

Now turning to the derivative with respect to Q_2:

$$\begin{aligned} \frac{\partial \pi}{\partial Q_2} &= R_2 \Pr(Q_2 < \min(D_2, q_2 + (Q_1 - D_1)^+)) \\ &\quad - s_1 \Pr((Q_1 - D_1)^+ > Q_2 - q_2, D_2 > Q_2) \\ &\quad + S_2 \Pr(Q_2 > \min(D_2, q_2 + (Q_1 - D_1)^+)) - C_2 \end{aligned} \tag{23}$$

This is equal to

$$\begin{aligned} \frac{\partial \pi}{\partial Q_2} &= R_2 - C_2 - (R_2 - S_2) \Pr(\min(D_2, q_2 + (Q_1 - D_1)^+) < Q_2) \\ &\quad - s_1 \Big(\Pr(D_2 > Q_2) - \Pr((Q_1 - D_1)^+ < Q_2 - q_2, D_2 > Q_2)\Big) \end{aligned} \tag{24}$$

By manipulation of the probability expressions, we get:

$$\begin{aligned} &\frac{\partial \pi}{\partial Q_2} = R_2 - C_2 - s_1 \\ &-(R_2 - S_2 - s_1)\Big(\Pr(D_2 < Q_2) + \Pr(q_2 + (Q_1 - D_1)^+ < Q_2 < D_2)\Big) \end{aligned} \tag{25}$$

Equating to zero and rearranging gives the first order condition:

$$\Pr(D_2 < Q_2) + \Pr(q_2 + (Q_1 - D_1)^+ < Q_2 < D_2) = \frac{R_2 - C_2 - s_1}{R_2 - S_2 - s_1} \tag{26}$$

Again, we find insight from this equation. Disregarding the second term on the left-hand side yields the optimal simple newsvendor solution for the case in which product 2 consists of its specific component and substitutable component 1. The numerator of the right-hand side, i.e. the shortage cost of not having enough of specific component 2, represents the lost revenue, R_2, less the cost of purchasing it, C_2, less the salvage value of sub component 1. (If the decision had been to increase Q_2 by one, we could have used another sub component 1, but now we can salvage that component instead.) The denominator is the overage cost plus the shortage cost. Therefore, the overage cost is $C_2 - S_2$. Buying too many of specific component 2 costs C_2, but loses the salvage value S_2. The second term on the left-hand side adjusts the simple newsvendor quantity *down* to account for the case in which product sales are constrained by the availability of sub components (i.e. the sum of q_2 and the (possible) leftovers of Q_1, and therefore of q_1).

Finally, the derivative with respect to q_2 is:

$$\begin{aligned}\frac{\partial \pi}{\partial q_2} &= R_2 \Pr(q_2 + (Q_1 - D_1)^+ < \min(D_2, Q_2)) \\ &\quad + s_1 \Pr(D_1 < Q_1, \min(D_2, Q_2) > q_2, Q_1 - D_1 > \min(D_2, Q_2) - q_2) \\ &\quad - S_2 \Pr(D_2 > q_2 + (Q_1 - D_1)^+, Q_2 > q_2 + (Q_1 - D_1)^+) \\ &\quad + s_2 \Pr(D_2 < q_2) - c_2 \qquad (27)\end{aligned}$$

Manipulating the probabilities, and using the fact from Property 1b that $Q_2^* \geq q_2^*$, this can be rewritten as:

$$\begin{aligned}\frac{\partial \pi}{\partial q_2} &= (R_2 - S_2 - c_2) - (R_2 - S_2 - s_2) \Pr(D_2 < q_2) \\ &\quad - (R_2 - S_2 - s_1) \Pr(q_2 < D_2 < q_2 + (Q_1 - D_1)^+ + (D_2 - Q_2)^+) \qquad (28)\end{aligned}$$

Equating to zero and rearranging gives the following first order condition:

$$\begin{aligned}&\Pr(D_2 < q_2) + \frac{R_2 - S_2 - s_1}{R_2 - S_2 - s_2} \Pr(q_2 < D_2 < q_2 + (Q_1 - D_1)^+ + (D_2 - Q_2)^+) \\ &= \frac{R_2 - S_2 - c_2}{R_2 - S_2 - s_2} \qquad (29)\end{aligned}$$

In this case, disregarding the second term on the left-hand side yields the optimal solution for the case in which product 2 consists only of the sub component and there is no substitution opportunity. The shortage cost (numerator) is the lost revenue of product 2, less the saved cost of purchasing the sub component and the salvage value of the specific component for product 2. The overage cost is $c_2 - s_2$, or the cost of the sub component less the salvage value we would have gained. The second term on the left-hand side adjusts the order quantity *down*, which accounts for two effects: (1) available units of the sub component of product 1, and (2) the specific component of product 2 being a constraint which might limit the usefulness of the sub component of product 2.

Clearly, the optimal order quantities are not easy to compute using the first order conditions. Therefore, one might want to search for the optimal values, or employ heuristics. To aid in these activities, we develop bounds for the optimal order quantities.

3.2 Bounds

In this section we develop bounds that can be computed by solving simple newsvendor problems. We shall see that these also provide insight into the optimal solutions as well as being a foundation for the future development of heuristics. Of course, they serve as guidelines for starting values in a search for the optimal values. We provide expressions for 5 out of the 6 possible bounds.

Property 4 *Let* Q_1^l *be the solution to* $\Pr(D_1 < Q_1) = \frac{R_1 - C_1 - c_1}{R_1 - S_1 - s_1}$*;* Q_1^l *is a lower bound of* Q_1^*.

Proof This follows directly from (22).□

Recall that by discarding the second term on the left-hand side of (22), we obtain a simple newsvendor problem, the right-hand side of which is the critical fractile for an independent product (product 1) with two components. The intuition is that if q_1 does not have an alternative use (i.e. can not be used to substitute for q_2) then we have a solution that is a lower bound of Q_1. As noted above, to obtain the optimal Q_1^*, the lower bound is adjusted up by the discarded term to account for the benefit of using the q_1 component as a substitute.

Property 5 *Let* Q_1^u *be the solution to* $\Pr(D_1 < Q_1) = \frac{R_1 - C_1 - c_1}{R_1 - S_1 - s_1 - (c_2 - s_2)}$; Q_1^u *is an upper bound of* Q_1^*.

Proof The derivative of the expected profit with respect to Q_1 can be written as:

$$\begin{aligned}\frac{\partial \pi}{\partial Q_1} &= R_1 - C_1 - c_1 - (R_1 - S_1 - s_1)\Pr(D_1 < Q_1) \\ &+ (R_2 - S_2 - s_1)\Pr(D_1 < Q_1, \min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2) \quad (30)\end{aligned}$$

By conditioning and unconditioning, this can be written as:

$$\begin{aligned}\frac{\partial \pi}{\partial Q_1} &= R_1 - C_1 - c_1 - (R_1 - S_1 - s_1)\Pr(D_1 < Q_1) \\ &+ (R_2 - S_2 - s_1)[\Pr(\min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2) \\ &\cdot \Pr(D_1 < Q_1 | \min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2)] \quad (31)\end{aligned}$$

Calling this difference caused by conditioning A,then:

$$A = \Pr(D_1 < Q_1) - \Pr(D_1 < Q_1 | \min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2)$$

Note that $A \geq 0$. The derivative of the expected profit with respect to q_2 can be written as:

$$\begin{aligned}\frac{\partial \pi}{\partial q_2} &= (R_2 - S_2 - s_1)\Pr(\min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2) \\ &+ (s_1 - s_2)\Pr(D_2 > q_2) + s_2 - c_2 \quad (32)\end{aligned}$$

Equating this to zero and rearranging, gives:

$$\begin{aligned}&(R_2 - S_2 - s_1)\Pr(\min(D_1, Q_1) + \min(D_2, Q_2) > Q_1 + q_2) \\ &= c_2 - s_2 - (s_1 - s_2)\Pr(D_2 > q_2) \quad (33)\end{aligned}$$

Substituting this into (31) gives:

$$\begin{aligned}\frac{\partial \pi}{\partial Q_1} &= R_1 - C_1 - c_1 - (R_1 - S_1 - s_1)\Pr(D_1 < Q_1) \\ &+ [[c_2 - s_2 - (s_1 - s_2)\Pr(D_2 > q_2)] \\ &\cdot (\Pr(D_1 < Q_1) - A)] \quad (34)\end{aligned}$$

Let

$$B = (c_2 - s_2)A + (s_1 - s_2)\Pr(D_2 > q_2)(\Pr(D_1 < Q_1) - A)$$

Note that $B \geq 0$. Using B and (34), we get:

$$\frac{\partial \pi}{\partial Q_1} = R_1 - C_1 - c_1 - (R_1 - S_1 - s_1 - c_2 + s_2)\Pr(D_1 < Q_1) - B \tag{35}$$

Equating to zero and rearranging gives:

$$\Pr(D_1 < Q_1) = \frac{R_1 - C_1 - c_1}{R_1 - S_1 - s_1 - (c_2 - s_2)} - \frac{B}{R_1 - S_1 - s_1 - (c_2 - s_2)} \tag{36}$$

This completes the proof.□

The intuition for the upper bound is similar to that of the lower bound. Discarding the second term on the right-hand side of (36) leaves a simple newsvendor problem. The underage cost of this simple problem is $R_1 - C_1 - c_1$, as before. The overage cost is $C_1 + c_1 - S_1 - s_1 - (c_2 - s_2)$, or the cost of buying too many components Q_1 and q_1 for product 1, less the overage cost for component q_2. The latter term is included because an additional q_1 could save us from buying an extra q_2. Again, to obtain the optimal order quantity from the upper bound, the simple newsvendor solution is adjusted *down* by the second term on the right-hand side of (36).

Property 6 *Let Q_2^l be the solution to $\Pr(D_2 < Q_2) = \frac{R_2 - C_2 - c_2}{R_2 - S_2 - s_2}$; Q_2^l is a lower bound of Q_2^*.*

Proof The derivative of the expected profit with respect to Q_2 can be written as:

$$\begin{aligned} \frac{\partial \pi}{\partial Q_2} &= (R_2 - S_2 - s_1)\Pr(Q_2 < \min(D_2, q_2 + (Q_1 - D_1)^+)) \\ &\quad + S_2 - C_2 \end{aligned} \tag{37}$$

The derivative of the expected profit with respect to q_2 can be written as:

$$\frac{\partial \pi}{\partial q_2} = (R_2 - S_2 - s_1)[1 - \Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+))$$

$$-\Pr(Q_2 < \min(D_2, q_2 + (Q_1 - D_1)^+))]$$

$$+(s_1 - s_2)\Pr(D_2 > q_2) + s_2 - c_2 \tag{38}$$

Equating (38) to zero and rearranging gives:

$$\begin{aligned}
&(R_2 - S_2 - s_1)\Pr(Q_2 < \min(D_2, q_2 + (Q_1 - D_1)^+)) = \\
&(R_2 - S_2 - s_1)[1 - \Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+))] \\
&+(s_1 - s_2)\Pr(D_2 > q_2) + s_2 - c_2
\end{aligned} \tag{39}$$

Substituting (39) into (37) gives:

$$\begin{aligned}
\frac{\partial \pi}{\partial Q_2} &= R_2 - C_2 - c_2 - (R_2 - S_2 - s_1)\Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+)) \\
&\quad -(s_1 - s_2)\Pr(D_2 < q_2)
\end{aligned} \tag{40}$$

Equating to zero and with some manipulation, this can be rewritten as

$$\begin{aligned}
&\Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+)) \\
&+\frac{s_1 - s_2}{R_2 - S_2 - s_2}\left[\Pr(D_2 < q_2) - \Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+))\right] \\
&= \frac{R_2 - C_2 - c_2}{R_2 - S_2 - s_2}
\end{aligned} \tag{41}$$

We know that

$$\Pr(D_2 < Q_2) \geq \Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+))$$

Also, we know that, using Property 1b,

$$\Pr(D_2 < q_2) - \Pr(D_2 < \min(Q_2, q_2 + (Q_1 - D_1)^+)) \leq 0$$

This completes the proof.□

The lower bound on Q_2 is directly analogous to that on Q_1. The underage cost of the simplified newsvendor problem is $R_2 - C_2 - c_2$, or the lost margin; and the overage cost is

$C_2 + c_2 - S_2 - s_2$, or the cost of buying too many components. These are the costs for an independent newsvendor problem for product 2. Introducing the possibility of substitution of q_2 will only increase the value of Q_2^* because there may be more "mates" for it by using q_1.

Property 7 *Let* Q_2^u *be the solution to* $\Pr(D_2 < Q_2) = \frac{R_2 - C_2 - s_1}{R_2 - S_2 - s_1}$*;* Q_2^u *is an upper bound of* Q_2^**.*

Proof This follows directly from (26).□

Here, the underage cost is $R_2 - C_2 - s_1$, which is the lost margin of not selling another unit of product 2, less the salvage value of the unit of q_1 that can now be gained because it is not used for product 2. The overage cost is $C_2 - S_2$, which is, of course, the cost of the unit of Q_2 less its salvage value.

Property 8 *Let* q_2^u *be the solution to* $\Pr(D_2 < q_2) = \frac{R_2 - S_1 - c_2}{R_2 - S_2 - s_2}$*;* q_2^u *is an upper bound of* q_2^**.*

Proof This follows directly from (29).□

The critical fracile of this bound has underage cost of $R_2 - S_1 - c_2$, which is the lost revenue of product 2, less the cost of buying the unit of q_2. This is further decreased by the salvage value of Q_1 that is now lost because of using a unit of q_1 in product 2. The overage cost is $c_2 + S_1 - S_2 - s_2$. An addtional unit of q_2 means that it must be bought (c_2), but we gain its salvage value (s_2) and that of a unit of Q_2. However, we lose the salvage value of Q_1.

4 Summary and Future Research

Our results are summarized in Table 1.

	Q_1	Q_2	q_1	q_1
First Order Conditions	(22)	(26)	(22)	(29)
Lower Bounds	Property 4	Property 6	Property 4	
Upper Bounds	Property 5	Property 7	Property 5	Property 8

Table 1: Results

The component substitution problem is notoriously difficult. Many researchers have addressed the end product case, but the component substitution case has not been addressed in any depth. We believe this absence is due to the complexity of the probability expressions involved. By utilizing techniques developed in previous work, we have been able to analyze this difficult problem. We have found expressions for the first order conditions on the optimal component order quantities; and we have developed five of the six possible bounds for these quantities.

Future research should continue to analyze this important problem by examining the multiperiod case and the N-product case. In addition, our bounds provide a good starting point for the development of heuristics. It would be useful to find and test simple-to-use heuristics.

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