THE RESIDUAL LIFE OF THE RENEWAL PROCESS: A SIMPLE ALGORITHM

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ABSTRACT

We develop a simple algorithm, that does not require convolutions, for computing the distribution of the residual life when the renewal process is discrete. We also analyze the algorithm for the particular case of lattice distributions.

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1 - Introduction

The residual life of a renewal process is defined as the time elapsed from some fixed time *t* until the following renewal. It is one of the random variables that describes the local behavior of a renewal process, the other being the age at time *t*, or the time since the last renewal. The residual life is widely used in modelling stochastic processes. In reliability theory it appears as the time until the next failure. In queueing theory, it is the remaining service time of the customer being served at the moment a customer arrives (see Gross & Harris (1985)); and in inventory theory, it is the undershoot of the reorder point when (*s*, *S*) policies are used (see Tijms (1976)).

Early work on the residual life includes that of Karlin (1958) who defines the residual life, or excess random variable, for a renewal process and presents its Laplace transform. He also describes the age, or deficit random variable. Karlin then presents the value of the residual life for the case of the exponential distribution. He applies the residual life to the case of the (*s*,*S*) inventory policy, but he restricts the application to exponential demands. (In the case of renewal processes with exponential interrnewal times, the residual life is also exponential.) Karlin also notes that the residual life and age of a renewal process are identical. Ross (1983) (pp. 63ff) discusses the residual life and age and notes their asymptotic behavior.

Although presented as a way of describing the local behavior of a stochastic process at a fixed time *t*, the expressions for the residual life, or for some of its parameters such as its expected value, generally assume that $t \to \infty$. In other words, asymptotic results are typically used. These results are clearly approximate for small *t*, but they are often adequate for the modelling purpose at hand. Unfortunately, this is not always so. In the context of the (*s*, *S*) inventory model, Baganha, Pyke & Ferrer (1994) show that there are cases for which the asymptotic approximation generates large errors. In particular, the approximation is not adequate for small values of $S - s$, low variance demand, and probability functions with few mass points. Some of these conditions occur, very often, in the context of multiechelon systems. To solve this problem we need either an approximation more suitable for these situations or a faster process to compute the exact distribution.

In this note we develop a simple recursive algorithm for computing the residual life of a renewal process when the renewal process is discrete. The algorithm is quite fast, can be developed on a spreadsheet in minutes, and does not require convolutions. It does, however, require knowledge of the distribution of the interrenewal times whereas the asymptotic approximation requires knowledge of only its first three moments.

The remainder of this paper is organized as follows: In Section 2 we discuss the renewal theoretic basis of the residual life distribution and its approximation. In Section 3 we present our algorithm for computing the residual life distribution, and in Section 4 we discuss applications of this material and in Section 5, we present a summary.

2 - The Residual Life

In this section we briefly discuss the residual life of a renewal process. We refer the reader for a more complete development to Heyman & Sobel (1982) (Chapter 5), and Ross (1983) (Chapter 3). We use the standard notation of renewal theory from Heyman & Sobel (1982) , with the addition of some problem-specific variables. Let $\{N(t); t \geq 0\}$ be a renewal process and define:

X_i	=	time between the (i-1)st and the ith renewal.
$F(x)$	=	cumulative distribution function (*cdf*) of the random variable X_i.
$f(x)$	=	density function or probability function of the random variable X_i.
$F_n(x)$	=	n-fold convolution of $F(x)$ with itself.
$f_n(x)$	=	n-fold convolution of $f(x)$ with itself.

$M(j)$ = $\sum_{n=1}^{\infty} F_n(j)$. = the renewal function, which can be thought of as the expected number of renewals at time j.

$m(j)$ = $\sum_{n=1}^{\infty} f_n(j)$ $j \geq 0$. Often called the renewal density. $m(j) * \Delta j$ represents the probability that a renewal occurs during $(j, j + \Delta j)$.

μ = expected value of X_i

σ^2 = variance of X_i.

$U(t)$ = the residual life at time t.

$h_t(u)$ = P($U(t)$ = u) = density or probability function of $U(t)$

$H_t(u)$ = distribution function or *cdf* of $U(t)$

When X_i is continuous $H_t(u)$ is given by the following equation (see Heyman & Sobel (1982), pp 131-132)

$$H_t(u) = F(t+u) - \int_0^t [1 - F(t+u-x)]dM(x)$$

If X_i is discrete we have:

$$H_t(u) = F(t+u) - \sum_{x=1}^{t-1} [1 - F(t+u-x)]m(x) \qquad (1)$$

and,

$$h_t(u) = f(t+u) - \sum_{x=0}^{t-1} f(t+u-x)m(x) \qquad (2)$$

When the inter-renewal time is exponential (continuous case) or geometric (discrete case) then $H_t(u)$ is also exponential or geometric and its distribution is independent of t. In all other cases the time until the next renewal is dependent on t. When using renewal theory it is generally assumed that the process has reached steady state; therefore, the asymptotic distribution is often used.

In the discrete case, the only case we will consider in this note, the limit of H_t only exists if the distribution is non-lattice. It is given by (Heyman & Sobel, p. 131)

$$\lim_{t\to\infty} H_t(u) = \frac{1}{\mu}\sum_{u=0}^{t-1}\left[1-F(u)\right] \tag{3}$$

Thus, the expected value and variance of *U(t)* as $t \to \infty$ are given by (Silver, Pyke & Peterson (1998) p. 333 or Hill (1988))

$$\lim_{t\to\infty} \mu_u = \frac{\sigma^2+\mu^2}{2\mu} - \frac{1}{2} \tag{4}$$

$$\lim_{t\to\infty} \sigma_u^2 = \frac{E\left(X^3\right)}{3\mu} - \left[\frac{\sigma^2+\mu^2}{2\mu}\right]^2 - \frac{1}{12} \tag{5}$$

In Baganha et al. (1994) we tested the robustness of these approximations, in the context of inventory theory, indicating conditions of good and poor performance. In cases of bad performance, one should use the exact distribution of the residual life. But normally this is computationally burdensome because of the convolutions necessary to obtain $m(x)$ and then to find $h_t(u)$. In the next section we present an algorithm to determine $h_t(u)$ without the use of convolutions.

3 - Computation of the Residual Life

Recall from elementary renewal theory that

$$m(j) = \sum_{i=1}^{\infty} f_i(j) \text{ for } j \geq 0$$

Using standard results of renewal theory, we have

$$m(j) = f(j) + \sum_{i=0}^{j} f(j-i)m(i)$$

Clearly, $m(0) = f(0) + f(0)m(0)$

and

$$m(0) = \frac{f(0)}{1-f(0)}. \tag{6}$$

So,

$$m(j) = f(j) + \sum_{i=0}^{j-1} f(j-i)m(i) + f(0)m(j)$$

From equation (2) we get

$$m(j) = h_j(0) + f(0)m(j)$$

Thus,

$$m(j) = \frac{h_j(0)}{1 - f(0)}. \tag{7}$$

Also by equation (2), we get

$$\begin{aligned} h_{t+1}(u-1) &= f(t+u) + \sum_{j=0}^{t} m(j)f(t+u-j) \\ &= f(t+u) + \sum_{j=0}^{t-1} m(j)f(t+u-j) + m(t)f(u) \\ &= h_t(u) + m(t)f(u) \end{aligned}$$

Substituting (7) implies

$$h_{t+1}(u-1) = h_t(u) + \frac{h_t(0)f(u)}{1 - f(0)}. \tag{8}$$

When t = 1, using equations (2) and (6) we obtain

$$h_1(u) = f(u+1) + m(0)f(u+1)$$

or

$$h_1(u) = \frac{f(u+1)}{1 - f(0)} \tag{9}$$

Using equations (8) and (9) we have the following algorithm, where u_{max} is the largest value of the residual life we are willing to consider, and T is the time since the process started. Thus, T is the desired value of t.

ALGORITHM

1. Let t = 1. Compute for all $u \le u_{max}$, :

$$h_1(u) = \frac{f(u+1)}{1 - f(0)}$$

   Go to Step 3.

2. Compute for all $u \le u_{max}$:

$$h_{t+1}(u-1) = h_t(u) + \frac{h_t(0)f(u)}{1-f(0)}.$$

3. Let $t = t + 1$. If $t \le T$ go to Step 2; otherwise Stop.

This algorithm can be easily implemented on a spreadsheet, requiring no more than $O(T{*}u_{max})$ basic operations. The choice of u_{max} is clear when the support of the renewal process is bounded. When $f(x)$ has an unbounded support the algorithm provides the exact distribution just for the range studied. In this case, given an ε as close to zero as desired, it is always possible to get a u_{max} such that

$$1 - H_T(u_{\max}) \le \varepsilon$$

Using the above results the expected value of the residual life may also be computed by means of a recursive equation. It is easy to show that:

$$\begin{aligned}
E[U(t+1)] &= \sum_{u=0}^{\infty} u h_{t+1}(u) \\
&= \sum_{u=0}^{\infty} u\left[h_t(u+1) + \frac{h_t(0)f(u+1)}{1-f(0)}\right] \\
&= \sum_{u=0}^{\infty} u h_t(u+1) + \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} u f(u+1) \\
&= \sum_{u=0}^{\infty} (u+1)h_t(u+1) + \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} (u+1) f(u+1) \\
&\qquad - \sum_{u=0}^{\infty} h_t(u+1) - \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} f(u+1) \\
&= E[U(t)] + \frac{h_t(0)}{1-f(0)} \mu - 1.
\end{aligned}$$

In a similar fashion we can compute $E(U^2(t))$, and from the first two moments, the variance of the residual life. We have thus defined a process that recursively computes the distribution of the residual life and its mean and variance.

The algorithm may be used to compute the distribution of the residual life for any value of *t*. If we want the limiting distribution we have to ensure that this algorithm will converge, i.e. that there is a *t* after which the distribution will not change.

Start by noticing that a corollary of the Key Renewal Theorem (see Ross(1983), p.65) is that

$$\lim_{t\to\infty} \frac{h_t(o)}{1-f(o)} = \frac{1}{\mu}$$

Based on this we get the convergence of the mean residual life if the conditions of the Key Renewal Theorem are fulfilled. One them being that the distribution *F* not be lattice.

The above algorithm will not converge if the distribution *F* of the interrenewal times is lattice. A distribution is lattice if the random variable *X*, with density *f*, is such that there exists $k \geq 0$ such that $\sum_{n=0}^{\infty} P(X = nk) = 1$, where *k* is the periodicity of f. Although, strictly speaking, any discrete distribution is lattice, the algorithm will not converge only when k > 1.

Interrenewal times having a lattice distribution with periodicity *k*, imply that renewals might only occur at moments that are multiples of *k*. Thus $m(j) = 0 \;\; for\; all \;\; j \neq nk,\, n \geq 1$ and integer. From this and expression (7) we get that $h_t(0) = 0 \;\; for \;\; all\; t \neq nk,\, n \geq 1$ and integer. Based on these results it is easy to prove the following proposition:

Proposition 1: *If f is lattice with positive probability only at values that are multiples of k, then* $h_t(u) = 0 \;\; for \;\; all\; t+u \neq nk,\, n \geq 1$ and integer.

This result shows that the distribution of the residual life will depend on the value of *t* + *u*. Thus there will not exist one but several limiting distributions depending on *t* + *u*..

Strictly speaking any discrete distribution is lattice. However in the particular case where *k* = 1, the limiting distribution exists, as the following proposition states (proof may be obtained from the authors:

Proposition 2: *If f is lattice with period k = 1, then* $\lim_{t\to\infty} h_t(u) - h_{t-1}(u) = 0$. *In other words, for any u,* $h_t(u)$ *converges as* $t \to \infty$.

4 – Applications

As mentioned in the introduction, the residual life of a renewal process is identical to the undershoot of the reorder point in an (*s*, *S*) inventory policy. In an (*s*, *S*) policy an order is placed when inventory position – on-hand plus on-order minus backorders – falls to or below the reorder point, *s*. The undershoot is the amount that inventory position is below the reorder point; and the order size is $S - s$ + the undershoot. Therefore the decision maker should have a reasonably accurate estimate of the size of the undershoot when determining the reorder point. Otherwise, safety stock will be inadequate for the desired service performance.

Our previous research suggests that there are cases, particularly if $S - s$ is small, when the asymptotic approximation is very poor. Because many manufacturing and logistics firms are approaching the just-in-time ideal of very small order sizes, we are seeing more and more cases of small values of $S - s$. Therefore, we recommend using our algorithm rather than the asymptotic approximation. In one consulting assignment, unfortunately completed prior to this research, we recommended using the approximate undershoot, and even delivered a spreadsheet to the client with the approximation built in. Since then we have tested the algorithm on some sample cases to see how much error is introduced. The results follow………

5 - Summary

In this paper we have developed a simple algorithm for computing the distribution of the residual life of discrete renewal processes. We also have provided a recursive formula to determine the expected value of the residual life as a function of cumulative time. We show that if the distribution of the time between renewals is lattice with period greater than 1, then the distribution of the residual life will not converge asymptotically. In these cases, the generally used approximation should be substituted by the algorithm that we have offered.

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