

# TECHNICAL APPENDIX TO MULTI-PRODUCT FIRMS AND PRODUCT SWITCHING

Andrew B. Bernard

*Tuck School of Business at Dartmouth  
National Bureau of Economic Research*

Stephen J. Redding

*London School of Economics and Yale  
Centre for Economic Policy Research*

Peter K. Schott

*Yale School of Management  
National Bureau of Economic Research*

January 2009

## A Introduction

In this technical appendix, we develop in further detail the model of multi-product firms and product switching discussed in the paper. The model builds closely on existing theories of industry dynamics, in which firms produce a single product and profitability varies stochastically across firms over time (see for example Jovanovic 1982, Hopenhayn 1992, Ericson and Pakes 1995, and Melitz 2003). In these existing models, firms that are heterogeneous in productivity are assumed to produce a single product, with the result that firm and product-market entry and exit are equivalent. Here, we develop a natural extension of such models in which firms choose to produce an endogenous range of products in response to evolving firm and product characteristics. The model highlights a number of features of product switching at the firm, product and firm-product level that we use to guide our empirical research and that are derived below.

The remainder of the technical appendix is structured as follows. Section B develops the model. Section C derives the features of product switching at the firm, product and firm-product level that we examine empirically in the paper. Section D characterizes the model's implications for revenue and quantity-based measures of productivity. For further economic intuition, Section E considers a particularly tractable special case of the model that abstracts from serial correlation in consumer tastes and productivity. Though this setup does not give rise to scale and age dependence in product dropping, it nevertheless provides insight into many of the other features of product switching found in the more general model and observed in our census data. An appendix at the end of this document contains technical derivations and the proofs of propositions.

## B Dynamic Firm-Product Selection Model

### B1. Preferences and Endowments

The representative consumer's utility is defined over the consumption of a fixed continuum of products, which we normalize to the interval  $[0, 1]$ . Utility is assumed to be a constant elasticity of substitution function of the consumption of each of the continuum of products:

$$U = \left[ \int_0^1 (a_i C_i)^\nu di \right]^{\frac{1}{\nu}}, \quad 0 < \nu < 1. \quad (1)$$

where  $i$  indexes products,  $a_i > 0$  is a demand parameter that allows the relative importance of products in utility to vary, and  $\kappa = \frac{1}{1-\nu} > 1$  is the elasticity of substitution between products.

As discussed further below, firms decide whether or not to participate in each product market, and if they choose to participate, to supply a horizontally differentiated variety of a product. Therefore  $C_i$  is a consumption index for product  $i$ , which is defined over the horizontally differentiated varieties of firms, and is also assumed to take the constant elasticity of substitution form:

$$C_i = \left[ \int_{\omega \in \Omega_i} (\lambda_i(\omega) c_i(\omega))^\rho d\omega \right]^{\frac{1}{\rho}}, \quad 0 < \rho < 1. \quad (2)$$

where  $\omega$  indexes firm varieties within product  $i$ . The demand parameter  $\lambda_i(\omega) \geq 0$  captures the representative consumer's tastes for a firm's variety within product  $i$  and allows demand to vary across the varieties of a given product supplied by different firms.<sup>1</sup> Although not central to our results, we make the natural assumption that the elasticity of substitution across varieties within products is greater than the elasticity of substitution across products,  $\sigma = \frac{1}{1-\rho} > \kappa = \frac{1}{1-\nu} > 1$ . Similarly, we also assume for simplicity that the elasticity of substitution across varieties within products,  $\sigma = \frac{1}{1-\rho}$ , is the same for all products. The price index dual to (2) is:

$$P_i = \left[ \int_{\omega \in \Omega_i} \left( \frac{p_i(\omega)}{\lambda_i(\omega)} \right)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}}. \quad (3)$$

Consumer expenditure minimization implies that revenue for a variety of a product  $r_i(\omega)$  depends upon the own variety price  $p_i(\omega)$ , consumer tastes  $\lambda_i(\omega)$ , the product price index  $P_i$  and aggregate product revenue  $R_i$ , while aggregate product revenue  $R_i$  depends on the product price indices  $P_i$  and aggregate revenue for the economy as whole  $R$ :

$$r_i(\omega) = p_i(\omega)^{1-\sigma} \lambda_i(\omega)^{\sigma-1} P_i^{\sigma-1} R_i, \quad R_i = \left[ \frac{(P_i/a_i)^{1-\kappa}}{\int_0^1 (P_i/a_i)^{1-\kappa} di} \right] R \quad (4)$$

Labor is the sole factor of production and is assumed to be in inelastic supply  $L$  (which also indexes the size of the economy).

---

<sup>1</sup>One interpretation of the parameter  $\lambda_i(\omega)$  is product quality, though it also captures other more subjective characteristics of a firm's variety that influence the representative consumer's demand for that variety.

## B2. Production Technology

A firm is defined by its production technology (productivity  $\varphi$ ) and its product attributes that influence demand (consumer tastes  $\lambda_i$ ).<sup>2</sup> Selection across firms is driven by heterogeneity in productivity  $\varphi$ , while selection within firms arises as a result of heterogeneity in consumer tastes across a firm's products  $\lambda_i$ . Productivity and consumer tastes evolve stochastically over time, and firms make endogenous decisions whether to enter and exit and whether to participate in individual product markets as productivity and consumer tastes evolve.

We assume that there is a competitive fringe of potential firms who are identical prior to entry. In order to enter, firms must incur a sunk entry cost of  $f_e > 0$  units of labor, which can be interpreted as an upfront investment in research and development (R&D). Incurring the sunk entry cost creates a firm brand and a blueprint for one horizontally differentiated variety of each product that can be produced using this brand. The firm's productivity and the demand for its variety of each product are uncertain prior to entry. Once the sunk cost has been incurred, the firm observes its productivity  $\varphi$  and consumer tastes for its products  $\lambda_i$ , and decides whether to enter or exit. If the firm decides to exit, the knowledge embodied in its productivity and product blueprints is lost, and the sunk cost must be incurred again in order for the firm to re-enter. If the firm decides to enter, productivity and consumer tastes evolve over time according to stochastic processes whose properties are known prior to entry.

We choose a specification for the evolution of productivity and consumer tastes that is both tractable and sufficiently general to match key features of the firm-product data. After the sunk entry cost is paid, firm productivity,  $\varphi \in [\underline{\varphi}, \bar{\varphi}] \subset [0, \infty)$ , is drawn from a continuous distribution  $g_e(\varphi)$ , and consumer tastes for the firm's variety,  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}] \subset [0, \infty)$ , are drawn from a continuous distribution  $z_{ei}(\lambda_i)$  for each product  $i$ . The subscript  $e$  is mnemonic for entry and the corresponding cumulative distributions are denoted by  $G_e(\varphi)$  and  $Z_{ei}(\lambda_i)$  respectively. To make the firm's problem an interesting one, we assume that the intervals for productivity and consumer tastes are sufficiently wide that we have an interior equilibrium, in which a firm's entry decision depends on its productivity and the decision to participate in individual product markets depends on consumer tastes. To make use of law of large numbers results, the productivity and consumer taste distributions are assumed to be independent across firms, which enables idiosyncratic risk to be perfectly diversified across firms. Similarly, we assume that the consumer taste distributions are independent across products and the consumer taste and firm productivity distributions are uncorrelated with one another, which simplifies the characterization of the equilibrium range of products produced by a firm.<sup>3</sup>

Following entry, a firm faces a constant Poisson probability  $\varepsilon_i$  of a shock to consumer tastes  $\lambda_i$  for its variety of product  $i$ , in which case a new value for consumer tastes  $\lambda'_i$  is drawn from a continuous conditional distribution  $z_{ci}(\lambda'_i|\lambda_i)$ . Both the probability of a consumer taste shock ( $\varepsilon_i$ ) and the conditional distribution for consumer tastes ( $z_{ci}(\lambda'_i|\lambda_i)$ ) are allowed to vary across products. Following entry, the firm also faces a constant Poisson probability  $\theta$  of a shock to productivity  $\varphi$ , in which case a new value for productivity  $\varphi'$  is drawn from a continuous conditional distribution  $g_c(\varphi'|\varphi)$ . The subscript  $c$  is mnemonic for conditional, and we denote

---

<sup>2</sup>We follow existing models of industry dynamics in taking a technological approach to the boundaries of the firm. As such we refrain from endogenizing the ownership of the knowledge assets that embody technology, as in the incomplete contracts literature following Grossman and Hart (1986). Implicitly, knowledge assets are assumed to be intangible so that they cannot be transferred beyond the boundaries of the firm using arms-length transactions.

<sup>3</sup>While the consumer taste and firm productivity distributions are independent of one another, there is interdependence in a firm's profitability across products, because firm productivity is common across products.

the corresponding cumulative distributions by  $Z_{ci}(\lambda'_i|\lambda_i)$  and  $G_c(\varphi'|\varphi)$  respectively. While the probabilities  $\varepsilon_i$  and  $\theta$  are independent across time, our formulation allows for serial correlation in consumer tastes and firm productivity, because the new draws for these variables following a stochastic shock depend on their existing values through the conditional distributions  $z_{ci}(\lambda'_i|\lambda_i)$  and  $g_c(\varphi'|\varphi)$ . We make the natural assumption of positive serial correlation,  $\partial Z_{ci}(\lambda'_i|\lambda_i)/\partial\lambda_i < 0$  and  $\partial G_c(\varphi'|\varphi)/\partial\varphi < 0$  for  $\lambda'_i \in [\underline{\lambda}, \bar{\lambda}]$  and  $\varphi' \in [\underline{\varphi}, \bar{\varphi}]$ , so that the probability of drawing a new value for consumer tastes (productivity) below  $\lambda'_i$  (below  $\varphi'$ ) is decreasing in the existing value of consumer tastes (productivity). In addition to stochastic shocks to productivity and consumer tastes, the firm faces a constant Poisson probability of death  $\delta$ , which captures *force majeure* events unrelated to the profitability of the firm.<sup>4</sup>

Once the sunk cost has been incurred and productivity and consumer tastes are observed, the firm decides whether to enter and which products to produce. We assume that there is a fixed corporate headquarters cost of  $f_h > 0$  units of labor, which the firm must incur irrespective of the number of products that it chooses to produce, and a fixed production cost of  $f_{pi} > 0$  units of labor for each product  $i$  that is produced. There is a constant marginal cost of production for each product, which depends upon the firm's productivity. Total labor employed by a firm with productivity  $\varphi$  is therefore:

$$l(\varphi) = f_h + \int_0^1 I_i \left[ f_{pi} + \frac{q_i(\varphi, \lambda_i)}{\varphi} \right] di, \quad (5)$$

where  $I_i$  is an indicator variable which equals one if a firm produces product  $i$  and zero otherwise, and  $q(\varphi, \lambda_i)$  denotes output of product  $i$  by a firm with productivity  $\varphi$  and demand parameter  $\lambda_i$ .

### B3. Firm-Product Profitability

Demand for a firm's variety of a product depends upon the own variety price, the price index for the product and the price indices for all other products. If a firm produces a product, it supplies only one of a continuum of varieties, and so is unable to influence the price index for the product. Additionally, the price of a firm's variety of one product only influences the demand for its varieties of other products through the price indices. Therefore, the firm's inability to influence the price indices implies that its profit maximization problem reduces to choosing the price of each product variety separately to maximize the profits derived from that product variety.<sup>5</sup> This optimization problem yields the standard result that the equilibrium price of a product variety is a constant mark-up over marginal cost:

$$p_i(\varphi, \lambda_i) = \frac{1}{\rho} \frac{w}{\varphi}, \quad (6)$$

where we choose the wage for the numeraire and so  $w = 1$ .

---

<sup>4</sup>The model can be extended to allow firms to make endogenous investments in improving productivity and enhancing consumer tastes, but these extensions are not central to the model's key predictions, which are driven by selection. Our focus on selection within firms mirrors the emphasis on selection across firms in existing industry dynamics models of firm creation and destruction, such as Hopenhayn (1992), Ericson and Pakes (1995) and Melitz (2003).

<sup>5</sup>The structure of our model eliminates strategic interaction within or between firms. This choice of model structure enables us to isolate the role of selection within and across firms from considerations of strategic interaction.

From consumer expenditure minimization, substituting for the pricing rule and using the choice of numeraire, equilibrium revenue and profits from a product variety are:

$$r_i(\varphi, \lambda_i) = R_i (\rho P_i \varphi \lambda_i)^{\sigma-1}, \quad \pi_i(\varphi, \lambda_i) = \frac{r_i(\varphi, \lambda_i)}{\sigma} - f_{pi}, \quad (7)$$

where  $R_i$  denotes aggregate expenditure on product  $i$ .

From equation (7), differences in firm productivity have exactly the same effects on equilibrium revenue and profits from a product as differences in consumer tastes for the firm's variety, because prices are a constant mark-up over marginal costs and demand exhibits a constant elasticity of substitution.<sup>6</sup> From the expression for equilibrium product revenue, the ratio of revenue for two varieties of the same product depends solely on the relative productivities of the firms producing those varieties and the relative strength of consumer demand for those varieties:

$$r_i(\varphi'', \lambda_i'') = (\varphi''/\varphi')^{\sigma-1} (\lambda_i''/\lambda_i')^{\sigma-1} r_i(\varphi', \lambda_i'). \quad (8)$$

The key economic decisions of a firm in the model are whether to enter or exit and in which product markets to participate. Under our assumptions, the firm's decision whether or not to produce a product takes a very tractable form. Fixed and marginal production costs for individual product varieties have no sunk component and consumer tastes for a firm's variety of a product evolve independently of whether the firm's variety is actually produced. The firm's decision whether or not to produce its variety of a product thus reduces to a period-by-period comparison of contemporaneous revenue and production costs.<sup>7</sup> The existence of product fixed production costs implies that there is a **zero-profit consumer taste cutoff**  $\lambda_i^*(\varphi)$  for a firm with productivity  $\varphi$ , such that the firm will produce the product if consumer tastes  $\lambda_i$  are equal to or greater than  $\lambda_i^*(\varphi)$ . From the expression for equilibrium profits above, the value of  $\lambda_i^*(\varphi)$  is defined by the following zero-profit condition:

$$r_i(\varphi, \lambda_i^*(\varphi)) = R_i (\rho P_i \varphi \lambda_i^*(\varphi))^{\sigma-1} = \sigma f_{pi}. \quad (9)$$

Using the expression for relative product variety revenue in equation (8) above, equilibrium revenue and profits from a product can be expressed relative to the revenue of a variety with the zero-profit consumer taste:

$$\begin{aligned} r_i(\varphi, \lambda_i(\varphi)) &= \begin{cases} \left(\frac{\lambda_i}{\lambda_i^*(\varphi)}\right)^{\sigma-1} \sigma f_{pi} & \text{for } \lambda_i \geq \lambda_i^*(\varphi) \\ 0 & \text{otherwise} \end{cases}, \\ \pi_i(\varphi, \lambda_i(\varphi)) &= \begin{cases} \left[\left(\frac{\lambda_i}{\lambda_i^*(\varphi)}\right)^{\sigma-1} - 1\right] f_{pi} & \text{for } \lambda_i \geq \lambda_i^*(\varphi) \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \quad (10)$$

As consumer tastes for a firm's variety of each product change over time, previously profitable products become unprofitable and are dropped when  $\lambda_i$  falls below the zero-profit cutoff

<sup>6</sup>Therefore  $\lambda_i$  has an equivalent interpretation as a component of a firm's productivity that is specific to individual products. Under this alternative interpretation, which leaves the determination of general equilibrium unchanged, stochastic shocks to  $\lambda_i$  capture changes in firm productivity that are specific to individual products.

<sup>7</sup>In contrast, the costs of firm entry are sunk, which introduces an option value to firm entry, as discussed further below. While our analysis can be extended to also allow for sunk costs of entering each product, the resulting option value to producing each product complicates the analysis without substantively changing the features of the model that we examine in our empirical work.

value  $\lambda_i^*(\varphi)$ . Similarly, previously unprofitable products become viable and are added when  $\lambda_i$  rises above  $\lambda_i^*(\varphi)$ .

The equilibrium of the model features an endogenous stationary distribution for consumer tastes  $\lambda_i$  across firms within each product  $i$ , which we denote by  $\gamma_{zi}(\lambda_i)$ , and which is influenced by both the entry and conditional distributions for consumer tastes and the probability of a consumer taste shock. The stationary distribution for consumer tastes is independent of firm productivity, since the entry and conditional distributions for consumer tastes are by assumption independent of the corresponding distributions for firm productivity. The stationary distribution of consumer tastes within each product is characterized by the requirement that the inflow of firms to consumer taste  $\lambda_i$  equals the corresponding outflow from consumer taste  $\lambda_i$  for all  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ :

$$z_{ei}(\lambda_i) [1 - G_e(\varphi^*)] M_e + \left[ \int_{\underline{\lambda}}^{\bar{\lambda}} \varepsilon_i z_{ic}(\lambda_i | \lambda'_i) \gamma_{zi}(\lambda'_i) d\lambda'_i \right] M = [\delta + \chi + \varepsilon_i] \gamma_{zi}(\lambda_i) M, \quad (11)$$

where  $M_e$  denotes the mass of entrants each period;  $M$  denotes the mass of firms producing;  $\varphi^*$  is the zero-value cutoff productivity below which a firm exits as determined below;  $\chi \equiv \theta \int_{\varphi^*}^{\bar{\varphi}} G_c(\varphi^* | \varphi) \gamma_g(\varphi) d\varphi$  is the aggregate probability of firm exit due to stochastic productivity shocks, which depends on the stationary distribution of firm productivity  $\gamma_g(\varphi)$  that is characterized below.

The equality of inflows and outflows in equation (11) has an intuitive interpretation. The first and second terms on the left-hand side are the inflow of firms to consumer taste  $\lambda_i$  as a result of firm entry and consumer taste shocks, respectively. The first, second and third terms on the right-hand side are respectively the outflow of firms from consumer taste  $\lambda_i$  as a result of firm death, productivity shocks that induce firm exit and consumer taste shocks. The stationary distribution for consumer tastes determined by the equality of inflows and outflows is defined over the interval  $[\underline{\lambda}, \bar{\lambda}]$ , such that  $\int_{\underline{\lambda}}^{\bar{\lambda}} \gamma_{zi}(\lambda_i) d\lambda_i = 1$ , and we denote the corresponding cumulative distribution by  $\Gamma_{zi}(\lambda_i)$ . There is also a corresponding stationary distribution for consumer tastes conditional on a product being produced, which is a truncation of the distribution  $\gamma_{zi}(\lambda_i)$  at the zero-profit cutoff  $\lambda_i^*(\varphi)$ .

#### B4. Firm Profitability

Having characterized a firm's decision whether to produce a product, we now examine firm profitability across the continuum of products as a whole. The total revenue and profits of a firm with productivity  $\varphi$  are as follows:

$$r(\varphi) = \int_0^1 I_i r_i(\varphi, \lambda_i) di, \quad \pi(\varphi) = \int_0^1 I_i \pi_i(\varphi, \lambda_i) di - f_h, \quad (12)$$

where  $I_i$  is again the indicator variable which equals one if a firm produces product  $i$  and zero otherwise.

With a continuum of products and independent distributions for consumer tastes, the law of large numbers implies that a firm's expected revenue across the continuum of products equals the sum of its expected revenues from each product. Total firm revenue and profits across the continuum of products are thus:

$$\begin{aligned}
 r(\varphi) &= \int_0^1 \left[ [1 - \Gamma_{zi}(\lambda_i^*(\varphi))] \int_{\lambda_i^*(\varphi)}^{\bar{\lambda}} r_i(\varphi, \lambda_i) \left( \frac{\gamma_{zi}(\lambda_i)}{1 - \Gamma_{zi}(\lambda_i^*(\varphi))} \right) d\lambda_i \right] di, \\
 \pi(\varphi) &= \int_0^1 \left[ [1 - \Gamma_{zi}(\lambda_i^*(\varphi))] \int_{\lambda_i^*(\varphi)}^{\bar{\lambda}} \pi_i(\varphi, \lambda_i) \left( \frac{\gamma_{zi}(\lambda_i)}{1 - \Gamma_{zi}(\lambda_i^*(\varphi))} \right) d\lambda_i \right] di - f_h.
 \end{aligned} \tag{13}$$

While consumer tastes for a firm's variety of a product are stochastic, the law of large numbers implies that all firms with the same productivity experience the same flow of total profits across the continuum of products. Stochastic shocks to consumer tastes generate fluctuations in the profitability of individual products, which lead them to be added and dropped over time. However, these fluctuations in the profitability of individual products average out at the level of the firm, so that the evolution of total firm profits over time is determined solely by stochastic shocks to firm productivity.

Once a firm observes its productivity and hence total profits across the continuum of products, it decides whether or not to enter. The lower a firm's productivity  $\varphi$ , the higher the zero-profit cutoff for product expertise  $\lambda_i^*(\varphi)$ , and so the lower the probability of having a value for consumer taste sufficiently high to profitably produce a product ( $1 - \Gamma_{zi}(\lambda_i^*(\varphi))$ ). With a continuum of products and independent distributions for consumer tastes, the fraction of products produced by a firm with productivity  $\varphi$  equals the sum of the probabilities of producing each product:

$$\Lambda(\varphi) = \int_0^1 (1 - \Gamma_{zi}(\lambda_i^*(\varphi))) di \tag{14}$$

As a firm with lower productivity has a lower probability of producing each product, it produces a smaller range of products than another firm with higher productivity. Therefore, the range of products produced by a firm is monotonically increasing in its productivity. Additionally, for sufficiently low values of a firm's productivity, the excess of revenue over fixed production costs in the small range of profitable products falls short of the fixed headquarters cost, and the instantaneous flow of total firm profits across the continuum of products is negative.

The existence of a sunk entry cost combined with stochastic shocks to productivity generates an option value to firm entry, so that a firm's entry decision does not simply depend on a comparison of total firm profits and the fixed headquarters cost. If a firm chooses to exit, it forgoes both the net present value of its instantaneous flow of profits and also the option of experiencing stochastic productivity shocks. Therefore, with this option value to firm entry, the threshold productivity for firm entry and exit will in general lie below the productivity at which the instantaneous flow of total firm profits is equal to zero. The value of a firm with productivity  $\varphi$  is determined according to the following Bellman equation:

$$v(\varphi) = \begin{cases} \frac{\pi(\varphi) + \theta \left[ \int_{\varphi^*}^{\varphi} [v(\varphi') - v(\varphi)] g_c(\varphi' | \varphi) d\varphi' \right]}{\delta + \theta G_c(\varphi^* | \varphi)} & \text{for } \varphi \geq \varphi^* \\ 0 & \text{otherwise} \end{cases}, \tag{15}$$

where  $\varphi^*$  is the zero-value cutoff productivity below which a firm exits;  $\delta$  is the exogenous probability of firm death;  $\theta G_c(\varphi^* | \varphi)$  is the endogenous probability of firm exit, which depends

on productivity  $\varphi$ , and equals the probability of a stochastic shock to productivity  $\theta$  times the probability of drawing a new value for productivity below the zero-value cutoff  $G(\varphi^*|\varphi)$ .

The Bellman equation (15) has an intuitive interpretation. The value of a firm with productivity  $\varphi$  is the flow of current profits plus the expected value of capital gains or losses as a result of a stochastic productivity shock, discounted by the probability of firm death. To determine the value of a firm with productivity  $\varphi$ , it proves convenient to re-write the Bellman equation in the following way:

$$v(\varphi) = \begin{cases} \frac{\pi(\varphi)}{\delta+\theta} + \frac{\theta}{\delta+\theta} \int_{\varphi^*}^{\bar{\varphi}} v(\varphi') g_c(\varphi'|\varphi) d\varphi' & \text{for } \varphi \geq \varphi^* \\ 0 & \text{otherwise} \end{cases}, \quad (16)$$

which, as shown in the appendix to this document, has the following solution:

$$v(\varphi) = \begin{cases} \frac{\pi(\varphi)}{\delta+\theta} + \frac{\theta}{\delta+\theta} \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') g_c(\varphi'|\varphi) d\varphi' & \text{for } \varphi \geq \varphi^* \\ 0 & \text{otherwise} \end{cases}, \quad (17)$$

where  $\eta(\varphi)$  is determined in the appendix.

The zero-value cutoff productivity  $\varphi^*$  below which exit occurs is defined by  $v(\varphi^*) = 0$ , which implies:

$$-\pi(\varphi^*) = \theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') g_c(\varphi'|\varphi^*) d\varphi', \quad (18)$$

so that at productivity  $\varphi^*$  the flow of current losses exactly equals the probability of a stochastic shock to productivity times expected profits following a productivity shock.

As well as an endogenous stationary distribution for consumer tastes within each product, the equilibrium of the model also features an endogenous stationary distribution for firm productivity  $\varphi$ , which we denote by  $\gamma_g(\varphi)$ . This stationary distribution for firm productivity is determined by the requirement that the inflow of firms to each productivity equals the corresponding outflow from each productivity for all  $\varphi \geq \varphi^*$ :

$$g(\varphi) M_e + \int_{\varphi^*}^{\bar{\varphi}} \theta g_c(\varphi|\varphi') \gamma_g(\varphi') d\varphi' M = (\delta + \theta) \gamma_g(\varphi) M \quad (19)$$

The equality of inflows and outflows from each productivity in equation (19) also has an intuitive interpretation. The first and second terms on the left-hand side are the inflow of firms to productivity  $\varphi$  from entry and productivity shocks, respectively. The first term on the right-hand side is the outflow of firms from productivity  $\varphi$  due to exogenous firm death. The second term on the right-hand side is the outflow of firms from productivity  $\varphi$  to other values of productivity as a result of productivity shocks, including productivity shocks that induce endogenous firm exit. We define the stationary distribution for firm productivity  $\gamma_g(\varphi)$  as conditional upon entry, so that  $\int_{\varphi^*}^{\bar{\varphi}} \gamma_g(\varphi) = 1$ , and we denote the corresponding cumulative distribution by  $\Gamma_g(\varphi)$ .

### B5. Firm Entry

The equilibrium zero-value cutoff productivity  $\varphi^*$  is determined by the free entry condition that requires the expected value of entry to equal the sunk entry cost. The expected value of entry is equal to the probability of successful entry times the expected value of the firm conditional on successful entry:

$$V = [1 - G_e(\varphi^*)] \bar{v} = f_e, \quad \bar{v} \equiv \int_{\varphi^*}^{\bar{\varphi}} v(\varphi) \left( \frac{g_e(\varphi)}{1 - G_e(\varphi^*)} \right) d\varphi, \quad (20)$$

where  $[1 - G_e(\varphi^*)]$  is the *ex ante* probability of drawing a productivity above the zero-value cutoff  $\varphi^*$ , and  $\bar{v}$  is the expected value of the firm conditional on entry, which depends on the equilibrium value of the firm for each productivity from (17) and the *ex ante* productivity distribution  $g_e(\varphi)$ .

Firms are assumed to finance the sunk entry cost by issuing equity to the representative consumer. As there is a continuum of firms, and the productivity and consumer tastes distributions are independent across firms, consumers can perfectly diversify the idiosyncratic risk of stochastic shocks to a firm's productivity and consumer tastes. Therefore the equilibrium rate of return received by the representative consumer on firm equity is equal to the aggregate risk of firm death and exit due to productivity shocks, which is equal to  $\delta + \chi \equiv \delta + \theta \int_{\varphi^*}^{\bar{\varphi}} G_c(\varphi^*|\varphi) \gamma_g(\varphi) d\varphi$ .

### B6. Goods and Labor Markets

While the analysis so far has characterized firm-level decisions about entry and production, we now turn to the determination of aggregate variables, such as the price indices and product revenue, that depend upon the mass of firms. The stationary equilibrium of the model is characterized by a constant mass of firms entering each period,  $M_e$ , and a constant mass of firms producing,  $M$ . Each of the firms producing is active in a subset of product markets, with the steady-state mass of firms producing a given product,  $M_{pi}$ , equal to a constant fraction of the mass of firms producing.

To determine the mass of firms producing a given product, we note that of the mass of firms with productivity  $\varphi$ , a fraction  $[1 - \Gamma_{zi}(\lambda_i^*(\varphi))]$  produce product  $i$ . Therefore the total mass of firms producing product  $i$  can be determined from this fraction by integrating over values for productivity using the stationary productivity distribution:

$$M_{pi} = \left[ \int_{\varphi^*}^{\bar{\varphi}} [1 - \Gamma_{zi}(\lambda_i^*(\varphi))] \gamma_g(\varphi) d\varphi \right] M, \quad (21)$$

The mass of firms producing,  $M$ , can be in turn determined from aggregate revenue,  $R$ , and the expected revenue of a firm conditional on production,  $\bar{r}$ :

$$M = \frac{R}{\bar{r}}, \quad \bar{r} = \int_{\varphi^*}^{\bar{\varphi}} r(\varphi) \gamma_g(\varphi) d\varphi \quad (22)$$

where revenue across the continuum of products for a firm with productivity  $\varphi$ ,  $r(\varphi)$ , is given by (13).

The mass of firms entering each period,  $M_e$ , can be determined from the mass of firms producing,  $M$ . With a constant mass of firms producing in the stationary equilibrium, the mass of firms that enter and draw a productivity above the zero-value cutoff must equal the

mass of firms that exit, which equals the mass of firms that die or draw a new productivity below the zero-value cutoff:

$$[1 - G_e(\varphi^*)] M_e = \left[ \delta + \theta \int_{\varphi^*}^{\bar{\varphi}} G_c(\varphi^*|\varphi) \gamma_g(\varphi) d\varphi \right] M. \quad (23)$$

The price indices for each product,  $P_i$ , can be determined from the mass of firms producing each product,  $M_{pi}$ . From the equilibrium pricing rule (6), the price index for a product can be written as a function of the mass of firms producing the product and the price of a variety with a weighted average of firm productivity and consumer tastes:

$$P_i = M_{pi}^{\frac{1}{1-\sigma}} \frac{1}{\rho \tilde{\varphi}_i}, \quad (24)$$

where the weighted average  $\tilde{\varphi}_i$  depends on the stationary distributions of firm productivity and consumer tastes and is defined in the appendix at the end of this document.

Aggregate revenue for each product,  $R_i$ , can be in turn determined from the price indices,  $P_i$ , and aggregate revenue,  $R$ , using equation (4). Finally, labor market clearing requires that the demand for labor in production and entry equals the economy's supply of labor:

$$L_q + L_e = \bar{L}, \quad (25)$$

where the subscripts  $q$  and  $e$  denote labor used in production and entry respectively.

### B7. General Equilibrium

General equilibrium is referenced by the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$ , together with a stationary distribution for firm productivity  $\gamma_g(\varphi)$  for  $\varphi \geq \varphi^*$ , and a stationary distribution for consumer tastes  $\gamma_{zi}(\lambda_i)$  for  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ . We begin by characterizing the stationary distributions of firm productivity and consumer tastes, before turning to consider the elements of the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$ .

#### B7.1. Stationary Distributions for Consumer Tastes and Productivity

The stationary distributions for firm productivity and consumer tastes are determined by the requirements of an equality of inflows to and outflows from each value of productivity and consumer tastes in equations (19) and (11). These equations imply that the stationary distributions of firm productivity and consumer tastes depend on the distributions from which these variables are drawn upon entry and the conditional distributions from which they are drawn following a stochastic shock. Since the entry and conditional distributions are continuous and have bounded support, there exist unique stationary distributions for firm productivity and consumer tastes, as long as the probabilities of a stochastic shock to consumer tastes and productivity do not exceed the thresholds specified in the proofs below.

**Proposition 1** *For  $\theta < \bar{\theta}$ , there exists a unique stationary distribution of firm productivity conditional upon entry,  $\gamma_g(\varphi)$ , for each value of the aggregate variables  $\{M_e, M, \varphi^*\}$ .*

**Proof.** See the appendix below. ■

**Proposition 2** *For  $\varepsilon_i < \bar{\varepsilon}$ , there exists a unique stationary distribution of consumer tastes,  $\gamma_{zi}(\lambda_i)$ , for each product  $i$  for each value of the aggregate variables  $\{M_e, M, \varphi^*\}$ .*

**Proof.** See the appendix below. ■

The stationary distributions of firm productivity and consumer tastes characterized in the proofs of the propositions are weighted averages of the entry and conditional distributions, with the weights depending on the equilibrium values of the mass of entrants  $M_e$ , the mass of firms  $M$ , the zero-value cutoff productivity below which firms exit  $\varphi^*$ , and the probabilities  $\theta$  and  $\varepsilon_i$  of a stochastic shock to firm productivity and consumer tastes respectively.

### B7.2. Zero-profit Consumer Taste Cutoffs

Having characterized the stationary distribution of firm productivity and consumer tastes, we now turn to consider the equilibrium value of the zero-profit consumer taste cutoff for each product,  $\lambda_i^*(\varphi)$ , which determines the equilibrium range of products produced by firms. As discussed above, the zero-profit consumer taste cutoff  $\lambda_i^*(\varphi)$  for each product varies across firms depending on their productivity  $\varphi$ . Using the expression for the relative revenues of product varieties in equation (8), the zero-profit consumer taste cutoff for each firm productivity can be expressed relative to that of a firm with the zero-value cutoff productivity:

$$\lambda_i^*(\varphi) = \left( \frac{\varphi^*}{\varphi} \right) \lambda_i^*(\varphi^*). \quad (26)$$

The above relationship has an intuitive interpretation. The zero-profit consumer taste cutoff  $\lambda_i^*(\varphi)$  is decreasing in firm productivity  $\varphi$  because the revenue derived from a product variety is increasing in firm productivity. Therefore, as a firm's productivity increases, it generates sufficient revenue to cover product fixed production costs at a lower value of consumer tastes. In contrast, the zero-profit consumer taste cutoff  $\lambda_i^*(\varphi)$  is increasing in the zero-value cutoff productivity  $\varphi^*$ . A higher value of  $\varphi^*$  increases the average productivity of rival firms, which intensifies product market competition and reduces product variety revenue, so that a higher value of consumer tastes is required to generate sufficient revenue to cover product fixed production costs. Similarly, an increase in the zero-profit consumer taste cutoff for the lowest productivity firm  $\lambda_i^*(\varphi^*)$  increases  $\lambda_i^*(\varphi)$  for all firm productivities, because it raises the average consumer tastes of rival firms, which intensifies product market competition and reduces product variety revenue.

The zero-profit consumer taste cutoff  $\lambda_i^*(\varphi^*)$  for a firm with the zero-value cutoff productivity  $\varphi^*$  on the right-hand side of (26) can be determined from the zero-profit cutoff condition (9) for productivity  $\varphi^*$ :

$$R_i (\rho P_i \varphi^* \lambda_i^*(\varphi^*))^{\sigma-1} = \sigma f_{pi}. \quad (27)$$

Together equations (26) and (27) determine the zero-profit consumer taste cutoff  $\lambda_i^*(\varphi)$  for firms of all productivities as a function of  $\{\varphi^*, P_i, R_i\}$ . Since the zero-profit consumer taste cutoffs  $\lambda_i^*(\varphi)$  determine the probability that each product is produced  $[1 - \Gamma_{zi}(\lambda_i^*(\varphi))]$ , these equations also determine the equilibrium range of products produced by firms of all productivities.

### B7.3. Zero-value Productivity Cutoff

To determine the zero-value cutoff productivity below which firms exit,  $\varphi^*$ , we use the free entry condition (20) that requires the expected value of entry to equal the sunk entry cost.

Substituting for the value of a firm from (17), the free entry condition can be re-written as follows:

$$V = \left( \frac{1}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} \left[ \pi(\varphi) + \theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') g_c(\varphi'|\varphi) d\varphi' \right] g_e(\varphi) d\varphi = f_e, \quad (28)$$

where the expected value of entry depends on the entry and conditional distributions for productivity and firm profits as a function of productivity.

#### B7.4. Product Price Indices and Revenues

The two remaining elements of the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$  are the price indices,  $P_i$ , and aggregate revenues,  $R_i$ , for each product. To characterize the equilibrium values of these variables for each product, we combine consumer and producer optimization with the free entry, steady-state stability and labor market clearing conditions.

We begin with the product price indices in equation (24), which depend on a weighted average of firm productivity and consumer tastes,  $\tilde{\varphi}_i$ , and the mass of firms that produce a product,  $M_{pi}$ . The weighted average,  $\tilde{\varphi}_i$ , can be determined from the zero-profit consumer taste cutoff,  $\lambda_i^*(\varphi)$ , the zero-value productivity cutoff,  $\varphi^*$ , and the stationary distributions for consumer tastes,  $\gamma_{zi}(\lambda_i)$ , and productivity,  $\gamma_g(\varphi)$ , as shown in the appendix. The mass of firms that produce a product,  $M_{pi}$ , is a constant fraction of the mass of active firms,  $M$ , as shown in equation (21). The mass of active firms,  $M$ , can be determined from aggregate economy-wide revenue,  $R$ , and average firm revenue conditional upon entry,  $\bar{r}$ , as shown in equation (22).

Average firm revenue conditional upon entry,  $\bar{r}$ , is solely a function of the zero-value productivity,  $\varphi^*$ , the zero-profit consumer taste cutoff for a firm with this productivity,  $\lambda_i^*(\varphi^*)$ , and the stationary distributions for firm productivity and consumer tastes,  $\gamma_{zi}(\lambda_i)$  and  $\gamma_g(\varphi)$ , as can be seen from equations (10), (13) and (26). Aggregate economy-wide revenue,  $R$ , can be determined by combining the free entry, steady-state stability and labor market clearing conditions. To do so, we solve for payments to labor used in entry and payments to labor used in production, and show that these sum to aggregate economy-wide revenue. Using the steady-state stability condition (23) to substitute for the probability of successful entry  $[1 - G_e(\varphi^*)]$  in the free entry condition (20), we obtain the following relationship between payments to labor used in entry and the expected value of the firm conditional on successful entry:

$$L_e = M_e f_e = [\delta + \chi] M \bar{v} \quad (29)$$

where  $\chi \equiv \theta \int_{\varphi^*}^{\bar{\varphi}} G_c(\varphi^*|\varphi) \gamma_g(\varphi) d\varphi$  is the aggregate probability of firm exit due to productivity shocks.

Equation (29) has an intuitive interpretation. Payments to labor used in entry on the left-hand side equal the flow rate of return received by the representative consumer on the average value of the firm conditional on entry on the right-hand side. Only when this condition is satisfied will the representative consumer break even on the equity issued by firms to finance the sunk costs of entry.

Payments to labor used in production equal aggregate economy-wide revenue minus the flow rate of return received by the representative consumer on the average equity value of the firm conditional on entry:

$$L_p = R - [\delta + \chi] M \bar{v} \quad (30)$$

Substituting the expressions for labor used in entry and production in equations (29) and (30) into the labor market clearing condition (25), aggregate economy-wide revenue equals the economy's labor endowment,  $R = L$ , and the labor market clears. Therefore, as we have characterized  $R$ ,  $\bar{r}$ ,  $M$ ,  $M_{pi}$  and  $\tilde{\varphi}_i$ , we have characterized the equilibrium price indices  $P_i$  in equation (24).

The final element of the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$  is aggregate product revenue  $R_i$ , which can be determined from consumer expenditure minimization, using the equilibrium price indices  $P_i$  and aggregate economy-wide revenue  $R$ :

$$R_i = \frac{R (P_i/a_i)^{1-\kappa}}{\int_0^1 (P_i/a_i)^{1-\kappa} di} \quad (31)$$

### B7.5. Existence and Uniqueness

The equilibrium quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$  and the stationary distributions for firm productivity  $\gamma_g(\varphi)$  and consumer tastes  $\gamma_{zi}(\lambda_i)$  are determined by the following six equations: the equality of inflows and outflows for productivity and consumer tastes ((19) and (11)), the free entry condition (28), the zero-profit consumer taste condition for the lowest productivity firm (27), the equilibrium price index (24), and equilibrium product revenue (31).

**Proposition 3** *For  $\theta < \bar{\theta}$  and  $\varepsilon_i < \bar{\varepsilon}$ , there exists a unique equilibrium referenced by the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$ , a stationary distribution for productivity  $\gamma_g(\varphi)$  for  $\varphi \geq \varphi^*$ , and a stationary distribution for consumer tastes  $\gamma_{zi}(\lambda_i)$  for  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ .*

**Proof.** See the appendix below. ■

The general equilibrium of the model features steady-state product switching by surviving firms and steady-state firm creation and destruction. Each period a measure of new firms incur the sunk entry cost. Of these new firms, those with a productivity draw above the zero-value cutoff enter, while those with a productivity draw below the zero-value cutoff exit. Among existing firms, a firm with unchanged productivity produces a constant range of products, but idiosyncratic shocks to consumer tastes for individual products induce the firm to drop a measure of the products previously produced and add an equal measure of products not previously produced. As stochastic shocks to an existing firm's productivity occur, the range of products produced expands with rises in productivity and contracts with declines in productivity. An existing firm exits endogenously when productivity falls below the zero-value cutoff or exogenously when death occurs as a result of force majeure considerations beyond the control of the firm.

## C Empirical Patterns of Product Switching

The model highlights a number of features of product switching at the firm, product and firm-product level, which we use to guide our empirical research. In this section, we show how these features of product switching can be derived from the model.

### C1. Firm-level Evidence

The model features steady-state adding and dropping of products within firms as a result of idiosyncratic shocks to the profitability of a firm's products. To determine the magnitude of steady-state product adding, consider the measure of products not currently produced by a firm with productivity  $\varphi$ , which depends on the stationary distribution of consumer tastes  $\gamma_{zi}(\lambda_i)$  over the interval  $\lambda_i \in [\underline{\lambda}, \lambda_i^*(\varphi)]$ . The probability that a product not currently produced is added by a firm with productivity  $\varphi$  equals the probability of drawing a new value for consumer tastes above the zero-profit cutoff  $\varepsilon_i(1 - Z_{ci}(\lambda_i^*(\varphi) | \lambda_i))$ . Integrating over products and the interval  $[\underline{\lambda}, \lambda_i^*(\varphi)]$ , the measure of products added by a firm with productivity  $\varphi$  is:

$$\Lambda^a(\varphi) = \int_0^1 \left[ \int_{\underline{\lambda}}^{\lambda_i^*(\varphi)} \varepsilon_i [1 - Z_{ci}(\lambda_i^*(\varphi) | \lambda_i)] \gamma_{zi}(\lambda_i) d\lambda_i \right] di. \quad (32)$$

Similarly, the measure of products dropped by a firm with productivity  $\varphi$  is:

$$\Lambda^d(\varphi) = \int_0^1 \left[ \int_{\lambda_i^*(\varphi)}^{\bar{\lambda}} \varepsilon_i Z_{ci}(\lambda_i^*(\varphi) | \lambda_i) \gamma_{zi}(\lambda_i) d\lambda_i \right] di. \quad (33)$$

As SIC codes can be interpreted as discrete partitions of the model's continuum of products, product adding in the data reflects firm production in a new five-digit SIC category, while product dropping in the data represents abandonment of production in one of the firms' existing five-digit SIC categories. Simultaneous adding and dropping of products occurs when both of these actions take place. The interpretation of industry and sector adding and dropping is analogous.

### C2. Product-level Evidence

One of the key features of patterns of product switching at the product level is a positive correlation between the rate of adding and dropping across products. To determine the measure of surviving firms that add and drop a product, we begin by considering the measure of firms with a particular value of productivity and consumer tastes for a product:  $\gamma_{zi}(\lambda_i) \gamma_g(\varphi) M$ . The probability that each of these firms survives after a unit interval of time depends on the probability of firm death,  $\delta$ , and the probability that a firm draws a new value for productivity below the zero-value cutoff,  $\theta G_c(\varphi^* | \varphi)$ . The measure of firms with a particular value of productivity and consumer tastes for a product that survive after a unit interval of time is therefore:

$$\xi_i(\varphi, \lambda_i) \equiv [1 - \delta - \theta G_c(\varphi^* | \varphi)] \gamma_{zi}(\lambda_i) \gamma_g(\varphi) M. \quad (34)$$

Among this measure of surviving firms with productivity  $\varphi$ , those with consumer tastes in the interval  $[\lambda_i^*(\varphi), \bar{\lambda}]$  currently produce the product. The probability that the product is dropped by each of these firms equals the probability of drawing a new value for consumer tastes below the zero-profit cutoff  $\varepsilon_i Z_{ci}(\lambda_i^*(\varphi) | \lambda_i)$ . Integrating over the interval of productivities for which firms enter and the interval of values for consumer tastes for which the product is produced, the total measure of surviving firms that drop the product after a unit interval of time is as follows:

$$\xi_i^d = \int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{\lambda_i^*(\varphi)}^{\bar{\lambda}} \varepsilon_i Z_{ci}(\lambda_i^*(\varphi) | \lambda_i) \xi_i(\varphi, \lambda_i) d\lambda_i \right] d\varphi. \quad (35)$$

Similarly, the total measure of surviving firms that add the product after a unit interval of time is as follows:

$$\xi_i^a = \int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{\underline{\lambda}}^{\lambda_i^*(\varphi)} \varepsilon_i [1 - Z_{ci}(\lambda_i^*(\varphi) | \lambda_i) \xi_i(\varphi, \lambda_i)] d\lambda_i \right] d\varphi. \quad (36)$$

Comparing equations (35) and (36), it is clear that “turbulent” products with high probabilities of idiosyncratic shocks  $\varepsilon_i$  are added and dropped more frequently, other things equal, than “stable” products with less frequent idiosyncratic shocks. Therefore, systematic differences across products in the degree of turbulence (the probability of idiosyncratic shocks  $\varepsilon_i$ ) induce a positive correlation between the rate at which a product is added and the rate at which it is dropped. As these idiosyncratic shocks lead some firms to add a product while other firms simultaneously drop the same product, the gross changes in output of a product as a result of product switching are systematically larger than the net changes in output.

### C3. Firm-Product Level Evidence

One of the central features of the model is selection within firms across products as well as selection across firms. The presence of selection within firms can be seen from the zero-profit cutoff condition for a product to be produced, which requires variable profits to cover the fixed production costs:

$$r_i(\varphi, \lambda_i^*(\varphi)) = (\rho P_i \varphi \lambda_i^*(\varphi))^{\sigma-1} = \sigma f_{pi}.$$

From this zero-profit condition, the decision whether to add or drop a product depends not only on characteristics of the firm (productivity  $\varphi$ ) but also on characteristics that are specific to the firm-product pair (consumer tastes  $\lambda_i$ ). Selection therefore operates within firms and depends on considerations that are idiosyncratic to individual firm-product pairs.

The process of selection within firms displays scale and age dependence as a result of the assumption that firm productivity and consumer tastes are positively serially correlated. Therefore a high existing value for productivity or consumer tastes reduces the probability of drawing a low new value of productivity or consumer tastes and dropping a product. As higher values of productivity and consumer tastes increase a firm’s shipments of a product (from equation (4)), the model implies that the probability a firm drops a product is decreasing in the firm’s shipments of the product (scale dependence). Additionally, as productivity and consumer tastes are serially correlated, the longer the length of time for which a firm has produced a product, the higher the firm’s expected productivity and consumer tastes, and hence the lower the probability of the firm dropping the product. Therefore the model also implies that the probability a firm drops a product is decreasing in the length of time for which the firm has produced the product (tenure dependence).<sup>8</sup>

---

<sup>8</sup>In the model, consumer tastes follow a first-order Markov process, so that the probability of drawing a new value for consumer tastes depends only on the current value of consumer tastes. Furthermore, controlling for firm and product fixed effects, log firm-product shipments are proportional to the current value of consumer tastes. Therefore, as in much of the firm entry and exit literature, age or tenure should become insignificant in a specification that controls appropriately for scale. One natural explanation for the significance of firm-product tenure in such a specification is that consumer tastes follow a higher-order Markov process, and the model could be extended to allow for this possibility.

## D Measured Quantity and Revenue-based Productivity

This section examines the relationship between the productivity draw in our model ( $\varphi$ ) and empirical measures of productivity used in the literature. Following Foster, Haltiwanger and Syverson (2008), we distinguish between quantity and revenue-based measures of productivity and show that both are different from, but monotonically related to,  $\varphi$ . We start with labor productivity, move on to index numbers of total factor productivity and conclude with productivity measures based on production function estimation. In each case, we first examine measured productivity for a product, before aggregating across products to derive measured productivity for the firm as a whole.

### D1. Measured Quantity-based Labor Productivity

For simplicity, our model assumes that labor is the sole factor of production and therefore yields direct implications for measured labor productivity. A firm's measured *quantity-based* labor productivity for *product*  $i$  ( $\theta_i^Q$ ) equals its quantity of output of the product per unit of labor employed:

$$\theta_i^Q \equiv \frac{q_i(\varphi, \lambda_i)}{l_i(\varphi, \lambda_i)} = \varphi \left( 1 - \frac{f_{pi}}{l_i(\varphi, \lambda_i)} \right), \quad (37)$$

where we have used  $q_i(\varphi, \lambda_i) = \varphi(l_i(\varphi, \lambda_i) - f_{pi})$ . From equation (37), measured quantity-based labor productivity for a product depends on the fixed and variable costs of production and equilibrium employment for the product.

Equilibrium employment for the product ( $l_i(\varphi, \lambda_i)$ ) is monotonically increasing in the productivity draw ( $\varphi$ ) given consumer tastes ( $\lambda_i$ ), as can be shown by combining the CES revenue function from equation (4) above, the equilibrium pricing rule from equation (6) above, and the production technology for the product from equation (5) above:

$$l_i(\varphi, \lambda_i) = \rho^\sigma (\varphi \lambda_i)^{\sigma-1} P_i^{\sigma-1} R_i + f_{pi}, \quad (38)$$

Together equations (37) and (38) can be used to characterize the relationship between measured quantity-based labor productivity for a product ( $\theta_i^Q$ ) and the firm productivity draw  $\varphi$ . We note that in the expression for equilibrium employment (38) product market conditions ( $P_i, R_i$ ) are the same for all firms producing the product. As a result measured quantity-based productivity for a product ( $\theta_i^Q$ ) only varies across firms because of variation in the firm productivity draw ( $\varphi$ ) and consumer tastes ( $\lambda_i$ ), where  $\lambda_i$  affects  $\theta_i^Q$  through equilibrium employment.

From equations (37) and (38), a firm's measured quantity-based labor productivity for a product is positively related to its productivity draw  $\varphi$  for two reasons. First, as  $\varphi$  rises, a given variable labor input generates more units of output. Second, as  $\varphi$  rises, variable labor input and output increase, and therefore the fixed labor input is spread over more units of output. At the *product* level, there is stochastic variation in variable labor input (and hence measured quantity-based labor productivity) for any given  $\varphi$ , because of stochastic variation in consumer tastes  $\lambda_i$ . However, consumer tastes and firm productivity are independently distributed, and consumer tastes are independently distributed across the unit continuum of products. Therefore, the distribution of consumer tastes across products is the same for each firm productivity, which implies that *firm* measured quantity-based labor productivity is monotonically related to  $\varphi$ .

Following standard empirical methods for productivity aggregation, we define measured productivity for the firm as a whole as the revenue-share weighted average of measured productivity for each product. Therefore measured quantity-based labor productivity for the firm as a whole is:

$$\Theta^Q \equiv \int_0^1 \theta_i^Q \frac{r_i(\varphi, \lambda)}{r(\varphi)} di, \quad (39)$$

### D2. Measured Revenue-based Labor Productivity

A firm's measured *revenue-based* labor productivity for *product*  $i$  ( $\theta_i^R$ ) equals its revenue from the product per unit of labor employed:

$$\theta_i^R \equiv \frac{p_i(\varphi, \lambda_i) q_i(\varphi, \lambda_i)}{l_i(\varphi, \lambda_i)} = \frac{1}{\rho} \left( 1 - \frac{f_{pi}}{l_i(\varphi, \lambda_i)} \right), \quad (40)$$

where we have used  $q_i(\varphi, \lambda_i) = \varphi (l_i(\varphi, \lambda_i) - f_{pi})$ , the equilibrium pricing rule in equation (6) above, and our choice of numeraire ( $w = 1$ ). From equation (40), measured revenue-based labor productivity for a product depends on the fixed costs of production and equilibrium employment for the product.

From equations (38) and (40), a firm's measured revenue-based labor productivity for a product is also positively related to  $\varphi$ , but now for only one of the two reasons discussed above. As  $\varphi$  rises, a given variable labor input generates more units of output, but the equilibrium pricing rule (6) implies that prices are inversely proportional to productivity. Therefore, as  $\varphi$  rises, the increase in the number of units of output generated by a given variable labor input is offset by proportionately lower prices, which leaves the revenue generated by a given variable labor input unchanged. Nonetheless, measured revenue-based labor productivity is still increasing in  $\varphi$ , because of the presence of fixed costs of production. As  $\varphi$  rises, variable labor input and revenue increase, and therefore the fixed labor input is spread over more units of revenue.

At the *product* level, there is again stochastic variation in variable labor input (and hence measured revenue-based productivity) for any given  $\varphi$ , because of stochastic variation in consumer tastes  $\lambda_i$ . However, as the distribution of consumer tastes across the unit continuum of products is the same for each firm productivity, *firm* measured revenue-based labor productivity is monotonically related to  $\varphi$ . Measured productivity for the firm as a whole is again defined as the revenue-share weighted average of measured productivity for each product:

$$\Theta^R \equiv \int_0^1 \theta_i^R \frac{r_i(\varphi, \lambda)}{r(\varphi)} di, \quad (41)$$

### D3. Relationship Between Quantity and Revenue-based Labor Productivity

The model features dispersion in both measured revenue and quantity-based labor productivity as a result of variation in productivity draws across firms. The two productivity measures are related at the product level as follows:

$$\theta_i^Q = \varphi \rho \theta_i^R.$$

Since measured revenue-based labor productivity ( $\theta_i^R$ ) is positively related to the firm productivity draw ( $\varphi$ ),  $\text{cov}(\varphi, \theta_i^R) > 0$ , which implies the following two results. First, measured

quantity and revenue-based labor productivity are positively correlated:  $\text{cov}(\theta_i^Q, \theta_i^R) > 0$ . Second, the variance of log measured quantity-based labor productivity is strictly greater than the variance of log measured revenue-based labor productivity:

$$\text{var}(\log \theta_i^Q) = \text{var}(\log \varphi) + \text{var}(\log \theta_i^R) + 2\text{cov}(\varphi, \theta_i^R) > \text{var}(\log \theta_i^R) > 0.$$

The intuition for this second result is that measured revenue-based labor productivity depends on both prices and measured quantity-based labor productivity, and prices are negatively correlated with measured quantity-based labor productivity, which implies that measured revenue-based labor productivity has a lower variance than measured quantity-based labor productivity.

While these results relate to a firm's measured quantity and revenue-based labor productivity for a product, they are consistent with the empirical findings for plants in Foster, Haltiwanger and Syverson (2008). Considering eleven products in the manufacturing sector for which separate data on plants' revenue and physical quantity of output are available, they find that measured quantity and revenue-based labor productivity are indeed positively correlated, and the variance of log measured quantity-based labor productivity is indeed greater than the variance of log measured revenue-based labor productivity.

The relationship between measured quantity and revenue-based labor productivity in our model is the same as that in the benchmark model of firm heterogeneity (Melitz 2003). As both models have constant elasticity of substitution preferences, prices are inversely proportional to the productivity draw  $\varphi$ , and dispersion in measured revenue-based productivity occurs because of fixed costs of production. In contrast, in other demand systems with variable elasticities of substitution, such as the quasi-linear preferences considered by Melitz and Ottaviano (2008), prices fall less than proportionately with the productivity draw  $\varphi$  because of endogenous changes in mark-ups. As a result dispersion in measured revenue-based productivity occurs because of fixed costs of production and/or endogenous changes in mark-ups.

#### *D4. Measured Total Factor Productivity*

While our model assumes for simplicity that labor is the sole factor of production, it can be extended to incorporate additional factors of production. For example, suppose that both the fixed and variable costs use two factors of production (labor and capital) according to a Cobb-Douglas technology.<sup>9</sup> Therefore the production technology for product  $i$  becomes:

$$f_{pi} + \frac{q_i(\varphi, \lambda_i)}{\varphi} = x_i(\varphi, \lambda_i),$$

where  $x_i(\varphi, \lambda_i)$  is a Cobb-Douglas composite factor input composed of labor and capital:

$$x_i(\varphi, \lambda_i) = l_i(\varphi, \lambda_i)^\vartheta k_i(\varphi, \lambda_i)^{1-\vartheta}, \quad 0 < \vartheta < 1.$$

Given this production technology for a product, standard index number measures of total factor productivity (TFP) can be constructed. Quantity and revenue-based measures of TFP can be defined, which are analogous to the measures of labor productivity in equations (37) and (40) above, but are based on the composite factor input ( $x_i(\varphi, \lambda_i)$ ) rather than labor input ( $l_i(\varphi, \lambda_i)$ ). Since the quantity and revenue-based measures of TFP take the same form as those for labor productivity above, the same analysis continues to apply.

---

<sup>9</sup>The production technology here takes the same form as that considered in the single-product firm model of Bernard, Redding and Schott (2007). While to simplify the exposition we consider two factors of production, the analysis generalizes in a straightforward way to more factors of production.

*D5. Productivity Measures from Production Function Estimation*

Quantity and revenue-based measures of productivity can also be constructed based on production function estimation. To simplify the exposition, we return to the case where labor is the sole factor of production. Assuming that the physical quantity of output and factor inputs can be observed by firm and product, and supposing that the functional form of the production technology is known and appropriate instruments for labor input are available, a firm's quantity-based productivity for a product can be estimated from the following regression:

$$\log q_i(\varphi, \lambda_i) = \log(l_i(\varphi, \lambda) - f_{pi}) + \log \varphi, \quad (42)$$

which yields the following measure of quantity-based productivity for the product:

$$\log \hat{\theta}_i^Q = \log \varphi. \quad (43)$$

Given data on revenue by firm and product, an analogous measure of revenue-based productivity can be estimated following the standard approach for production function estimation using revenue data in differentiated product markets (see for example Klette and Griliches 1996, Levinsohn and Melitz 2006, and De Loecker 2008). Deflating a firm's revenue for product  $i$  by the aggregate price index for that product,  $P_i$ , yields:

$$\tilde{r}_i(\varphi, \lambda_i) \equiv \frac{r_i(\varphi, \lambda_i)}{P_i} = \frac{p_i(\varphi, \lambda_i) q_i(\varphi, \lambda_i)}{P_i},$$

Using the inverse CES demand curve to substitute for  $p_i(\varphi, \lambda_i)$  on the right-hand side in terms of  $q_i(\varphi, \lambda_i)$ , we obtain:

$$\tilde{r}_i(\varphi, \lambda_i) = \lambda_i^{\frac{\sigma-1}{\sigma}} q_i(\varphi, \lambda_i)^{\frac{\sigma-1}{\sigma}} \left( \frac{R_i}{P_i} \right)^{\frac{1}{\sigma}},$$

Using the production technology ( $q_i(\varphi, \lambda_i) = \varphi(l_i(\varphi, \lambda_i) - f_{pi})$ ) and taking logarithms, we obtain the following "revenue production function" for a product:

$$\log \tilde{r}_i(\varphi, \lambda_i) = \frac{\sigma-1}{\sigma} \log(l_i(\varphi, \lambda) - f_{pi}) + \frac{1}{\sigma} \log \left( \frac{R_i}{P_i} \right) + \frac{\sigma-1}{\sigma} \log(\varphi \lambda_i), \quad (44)$$

where with constant elasticity of substitution demand, aggregate product market conditions ( $R_i/P_i$ ) shift the revenue production function for each firm equi-proportionately.

Assuming that the functional form of the revenue production function is known, and given appropriate instruments for labor input and controls for aggregate product market conditions, a firm's revenue-based productivity for a product can be estimated from the regression (44):

$$\log \hat{\theta}_i^R = \left( 1 - \frac{1}{\sigma} \right) \log(\varphi \lambda_i), \quad \sigma > 1. \quad (45)$$

From equation (45), as is standard in the production function estimation literature, productivity measures based on revenue rather than physical quantity data incorporate demand ( $\lambda_i$ ) as well as technical efficiency ( $\varphi$ ).

The measures of productivity from the production function estimation again feature dispersion in both revenue and quantity-based productivity. The two measures of a firm's productivity for a product are related as follows:

$$\log \hat{\theta}_i^R = \left( 1 - \frac{1}{\sigma} \right) \log \hat{\theta}_i^Q + \left( 1 - \frac{1}{\sigma} \right) \log \lambda_i, \quad (46)$$

which implies that measured revenue-based productivity is again positively correlated with measured quantity-based productivity. Furthermore, as the distributions of firm productivity and consumer tastes are assumed to be independent of one another, we obtain the following result for the two measures of a firm's productivity for a product:

$$\text{var} \left( \log \hat{\theta}_i^R \right) = \left( 1 - \frac{1}{\sigma} \right)^2 \text{var} \left( \log \hat{\theta}_i^Q \right) + \left( 1 - \frac{1}{\sigma} \right)^2 \text{var} (\log \lambda_i).$$

Since  $0 < 1 - \frac{1}{\sigma} < 1$  and  $\text{var}(\log \lambda_i) > 0$ , measures of revenue-based productivity from the production function estimation can exhibit more or less dispersion at the *product* level than those of quantity-based productivity depending on the elasticity of substitution and the relative variance of consumer tastes and technical efficiency.<sup>10</sup>

As consumer tastes and firm productivity are independently distributed, and consumer tastes are independently distributed across the unit continuum of products, the consumer tastes distribution is the same for each firm productivity. Therefore, while stochastic variation in consumer tastes induces stochastic variation in measured revenue-based productivity at the *product* level, measured *firm* revenue and quantity-based productivity are monotonically related to the productivity draw  $\varphi$ . Integrating over the continuum of products in equation (45), the term in consumer tastes ( $\lambda_i$ ) becomes a constant. As a result the variance of *firm* measured revenue-based productivity is strictly less than the variance of firm measured quantity-based productivity, since  $0 < 1 - \frac{1}{\sigma} < 1$ .

## E A Simple Special Case

To provide further economic intuition for the workings of the model, this section considers a simple special case of the more general framework developed above. This special case makes a number of simplifying assumptions in order to achieve greater tractability and derive closed form solutions. The most important of these simplifications is to abstract from serial correlation in firm productivity and consumer tastes. While this abstraction implies that the special case of the model does not capture scale and tenure dependence in the probability that a product is dropped by a firm, it remains consistent with the other features of product switching examined in our empirical work.

The additional simplifying assumptions made in the special case of the model are as follows: (a) all products  $i$  receive the same weight in consumer utility ( $a_i = 1$  for all  $i$ ), though consumer tastes for a firm's variety  $\omega$  of each product still vary (in general  $\lambda_i(\omega'') \neq \lambda_i(\omega')$  for  $\omega'' \neq \omega'$ ); (b) all products have the same fixed production costs ( $f_{pi} = f_p$  for all  $i$ ); (c) the probability of a stochastic shock to consumer tastes is the same across products ( $\varepsilon_i = \varepsilon$  for all  $i$ ); (d) the distribution from which consumer tastes are drawn following a stochastic shock is the same as upon entry ( $z_{ei}(\lambda_i) = z_{ci}(\lambda_i|\lambda'_i) = z(\lambda_i)$  for all  $i$  and  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ ); and (e) the distribution from which firm productivity is drawn following a stochastic shock is the same as upon entry ( $g_e(\varphi) = g_c(\varphi|\varphi') = g(\varphi)$  for  $\varphi \in [\underline{\varphi}, \bar{\varphi}]$ ).

Since the distributions from which consumer tastes and productivity are drawn following stochastic shocks are the same as the distributions from which they are drawn upon entry, the

---

<sup>10</sup> A key reason for this difference between the measures of productivity based on production function estimation and those considered above is that the coefficient on labor input is allowed to vary between the regressions with physical quantity and revenue data in equations (42) and (44) respectively.

stationary distributions for consumer tastes and productivity take a particularly simple form. The stationary distribution for consumer tastes is the same as the distribution upon entry:

$$\gamma_{zi}(\lambda_i) = z(\lambda_i), \quad (47)$$

while the stationary distribution for consumer tastes conditional on a product being produced is a truncation of  $z(\lambda_i)$  at the zero-profit cutoff for consumer tastes  $\lambda_i^*(\varphi)$ .

The stationary distribution for firm productivity conditional upon entry is a truncation of the distribution  $g(\varphi)$  at the zero-value cutoff productivity below which firms exit:

$$\gamma_g(\varphi) = \begin{cases} \frac{g(\varphi)}{1-G(\varphi^*)} & \text{for } \varphi \geq \varphi^* \\ 0 & \text{otherwise} \end{cases}. \quad (48)$$

With symmetric products and independently and identically distributed consumer tastes, the total profits of a firm with productivity  $\varphi$  across the unit continuum of products equal its expected profits from each product:

$$\pi(\varphi) = \int_{\lambda^*(\varphi)}^{\bar{\lambda}} \left( \left( \frac{\lambda}{\lambda^*(\varphi)} \right)^{\sigma-1} - 1 \right) f_p z(\lambda) d\lambda - f_h. \quad (49)$$

where we have used  $r(\varphi, \lambda^*(\varphi)) = \sigma f_p$  and  $r(\varphi, \lambda) = (\lambda/\lambda^*(\varphi))^{\sigma-1} r(\varphi, \lambda^*(\varphi))$ .

A firm with productivity  $\varphi$  draws a consumer taste above the zero-profit cutoff  $\lambda^*(\varphi)$  with probability  $[1 - Z(\lambda^*(\varphi))]$ , and so the range of products produced by a firm with productivity  $\varphi$  equals  $[1 - Z(\lambda^*(\varphi))]$ , which is monotonically increasing in  $\varphi$ . Firms with higher productivity generate sufficient revenue to cover the fixed costs of producing a product at a lower value for consumer tastes, and therefore manufacture a larger range of products than firms with lower productivity.

The Bellman equation for the value of a firm with productivity  $\varphi$  takes the same form as above:

$$v(\varphi) = \frac{\pi(\varphi)}{\delta + \theta} + \left( \frac{\theta}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} v(\varphi') g(\varphi') d\varphi', \quad (50)$$

Since the distribution from which a new value of productivity is drawn following a stochastic shock is the same as upon entry and is independent of a firm's existing value of productivity, the solution to the Bellman equation takes a particularly simple form in this special case of the model. Substituting for  $v(\varphi')$  on the right-hand side of (50) using the trial solution  $v(\varphi) = \alpha\pi(\varphi) + \beta \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi') g(\varphi') d\varphi'$ , and solving for  $\alpha$  and  $\beta$ , yields the equilibrium value of a firm:

$$v(\varphi) = \frac{\pi(\varphi)}{\delta + \theta} + \left( \frac{\theta}{\delta + \theta} \right) \left( \frac{1}{\delta + \theta G(\varphi^*)} \right) \left[ \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi') g(\varphi') d\varphi' \right], \quad (51)$$

Therefore the equilibrium value of a firm with productivity  $\varphi$  is a weighted average of the current flow of firm profits and the expected flow of firm profits following a stochastic productivity shock, where the weights depend on the probability of firm death, the probability of a productivity shock and the probability that a firm remains active following a productivity shock. Substituting the expression for the equilibrium value of a firm into the free entry condition, and rearranging, the free entry condition can be re-written as follows:

$$V = [1 - G(\varphi^*)] \int_{\varphi^*}^{\bar{\varphi}} \left[ \frac{\pi(\varphi) + \left( \frac{\theta}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} [\pi(\varphi') - \pi(\varphi)] g(\varphi') d\varphi'}{\delta + \theta G(\varphi^*)} \right] \left( \frac{g(\varphi)}{1 - G(\varphi^*)} \right) d\varphi = f_e, \quad (52)$$

which can be simplified to yield the following expression:

$$V = \int_{\varphi^*}^{\bar{\varphi}} \left( \frac{\pi(\varphi)}{\delta + \theta G(\varphi^*)} \right) g(\varphi) d\varphi = f_e, \quad (53)$$

where we have used  $\int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi') g(\varphi') d\varphi' \right] g(\varphi) d\varphi = \int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi) g(\varphi') d\varphi' \right] g(\varphi) d\varphi = [1 - G(\varphi^*)] \left[ \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi') g(\varphi') d\varphi' \right] = [1 - G(\varphi^*)] \left[ \int_{\varphi^*}^{\bar{\varphi}} \pi(\varphi) g(\varphi) d\varphi \right]$ , which implies that the second term in the numerator of the expression inside the square parentheses in equation (52) is equal to zero.

The expression for the expected value of entry in the free entry condition (53) has an intuitive interpretation. In the special case of the model considered here, the distribution from which a new value of productivity is drawn following a stochastic shock is the same as upon entry and independent of a firm's existing value of productivity. Therefore the expected value of entry is equal to expected firm profits discounted by the sum of the probability of firm death and the probability that a firm remains active following a productivity shock.

Using the expression for firm profits from equation (49), and noting from the analysis of the general model above that  $\lambda^*(\varphi) = (\varphi^*/\varphi) \lambda^*(\varphi^*)$ , the free entry condition (53) can be written solely in terms of parameters and two unknowns: the zero-value cutoff productivity  $\varphi^*$  and the zero-profit consumer taste cutoff for a firm with this productivity  $\lambda^*(\varphi^*)$ ,

$$V = \left( \frac{1}{\delta + \theta G_c(\varphi^*)} \right) \int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{(\varphi^*/\varphi)\lambda^*(\varphi^*)}^{\bar{\lambda}} \left( \left( \frac{\varphi}{\varphi^*} \frac{\lambda}{\lambda^*(\varphi^*)} \right)^{\sigma-1} - 1 \right) f_p z(\lambda) d\lambda - f_h \right] g(\varphi) d\varphi = f_e. \quad (54)$$

The free entry condition (54) provides one of two equations that together pin down the zero-value cutoff productivity  $\varphi^*$  and the zero-profit consumer taste cutoff for a firm with this productivity  $\lambda^*(\varphi^*)$ . The second of the two equations is obtained from the requirement that the value of a firm with the zero-value cutoff productivity  $\varphi^*$  is equal to zero:  $v(\varphi^*) = 0$ . Using the expression for firm profits from (49) in the expression for the equilibrium value of a firm in (51), the requirement that  $v(\varphi^*) = 0$  can be written as follows:

$$\begin{aligned} & - \left[ \int_{\lambda^*(\varphi^*)}^{\bar{\lambda}} \left( \left( \frac{\lambda}{\lambda^*(\varphi^*)} \right)^{\sigma-1} - 1 \right) f_p z(\lambda) d\lambda - f_h \right] \\ & = \left( \frac{\theta}{\delta + \theta G(\varphi^*)} \right) \int_{\varphi^*}^{\bar{\varphi}} \left[ \int_{(\varphi^*/\varphi)\lambda^*(\varphi^*)}^{\bar{\lambda}} \left( \left( \frac{\varphi}{\varphi^*} \frac{\lambda}{\lambda^*(\varphi^*)} \right)^{\sigma-1} - 1 \right) f_p z(\lambda) d\lambda - f_h \right] g(\varphi) d\varphi \end{aligned} \quad (55)$$

The free entry condition (54) is monotonically decreasing in  $\varphi^*$  for a given value of  $\lambda^*(\varphi^*)$ . Similarly, for a given value of  $\varphi^*$ , the left-hand side of equation (55) is monotonically increasing in  $\lambda^*(\varphi^*)$ , while the right-hand side of equation (55) is monotonically decreasing in  $\lambda^*(\varphi^*)$ . Therefore equations (54) and (55) together determine unique equilibrium values of  $\{\varphi^*, \lambda^*(\varphi^*)\}$ .

In the special case of the model, general equilibrium is again referenced by the quadruple  $\{\varphi^*, \lambda_i^*(\varphi^*), P_i, R_i\}$ , together with a stationary distribution for firm productivity  $\gamma_g(\varphi)$  for  $\varphi \geq \varphi^*$ , and a stationary distribution for consumer tastes  $\gamma_{zi}(\lambda_i)$  for  $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ . The stationary distributions of consumer tastes and firm productivity  $\{\gamma_z(\lambda), \gamma_g(\varphi)\}$  were determined above in equations (47) and (48). The equilibrium values of  $\{\varphi^*, \lambda^*(\varphi^*)\}$  can be determined from equations (54) and (55) as discussed above. Finally, the equilibrium values of the price

indices,  $P_i = P$ , and product revenue,  $R_i = R$ , can be determined in the same way as for the general model in section B7.4. above. Therefore this completes the characterization of general equilibrium in the special case of the model.

Despite its simplifying assumptions, the special case of the model still captures a number of the key features of product switching examined in our empirical work. Steady-state product switching is driven by idiosyncratic shocks to consumer tastes, while steady-state firm creation and destruction is the result of stochastic shocks to firm productivity. The special case of the model features selection both within and across firms. There is selection within firms because firms have systematically higher values of consumer tastes in the products that they choose to produce than in the products they choose not to produce. There is selection across firms because firms that draw values of productivity below the zero-value cutoff  $\varphi^*$  exit immediately without producing.

In the special case of the model, however, a firm's new values for consumer tastes and productivity following a stochastic shock are by assumption uncorrelated with its existing values:  $g_e(\varphi) = g_c(\varphi|\varphi') = g(\varphi)$  and  $z_{ei}(\lambda_i) = z_{ci}(\lambda_i|\lambda'_i) = z(\lambda_i)$ . Therefore the special case of the model does not capture the empirical finding that surviving firms that drop a product have systematically smaller shipments of the product than surviving firms that retain the product. To capture this empirical finding, one requires serial correlation in consumer tastes, as in the general model, so that  $z_{ei}(\lambda_i) \neq z_{ci}(\lambda_i|\lambda'_i)$ . The special case of the model also does not capture the empirical finding that exiting firms have smaller total shipments than surviving firms, because *conditional upon entry* the probability that a firm subsequently exits is independent of total shipments. To capture this second empirical finding, one requires serial correlation in productivity, as in the general model, so that  $g_e(\varphi) \neq g_c(\varphi|\varphi')$ .

## F Appendix to Technical Appendix

### F1. Weighted Average of Firm Productivity and Consumer Tastes

The aggregate weighted-average of firm productivity and consumer tastes is:

$$\tilde{\varphi}_i \equiv \left[ \int_{\varphi^*}^{\bar{\varphi}} \tilde{\lambda}_i(\varphi) \gamma_g(\varphi) d\varphi \right]^{\frac{1}{\sigma-1}}, \quad (56)$$

where the weights are given by the stationary distribution of firm productivity conditional upon entry  $\gamma_g(\varphi)$ . The function  $\tilde{\lambda}_i(\varphi)$  is a weighted-average of consumer tastes for a firm with productivity  $\varphi$ :

$$\tilde{\lambda}_i(\varphi) = \frac{1}{1 - \Gamma_{zi}(\lambda_i^*(\varphi))} \int_{\lambda_i^*(\varphi)}^{\bar{\lambda}} (\varphi \lambda_i)^{\sigma-1} \gamma_{zi}(\lambda_i) d\lambda_i, \quad (57)$$

where the weights given by the stationary distribution of consumer tastes conditional on a product being produced  $\gamma_{zi}(\lambda_i) / [1 - \Gamma_{zi}(\lambda_i^*(\varphi))]$ , and where  $\lambda_i^*(\varphi) = (\varphi^*/\varphi) \lambda_i^*(\varphi^*)$ .

### F2. Derivation of the Solution to the Bellman Equation in the General Model

Consider the following trial solution to the Bellman equation (16):

$$v(\varphi) = \alpha \pi(\varphi) + \beta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') g_c(\varphi'|\varphi) d\varphi', \quad (58)$$

where  $\alpha$ ,  $\beta$  and  $\eta(\varphi)$  are parameters and functions to be determined below. Using the trial solution to substitute for  $v(\varphi')$  on the right-hand side of the Bellman equation (16) gives:

$$v(\varphi) = \frac{\pi(\varphi)}{\delta + \theta} + \left( \frac{\theta}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} \left[ \frac{\alpha \pi(\varphi') + \beta \int_{\varphi^*}^{\bar{\varphi}} \eta(\psi) \pi(\psi) g_c(\psi|\varphi') d\psi}{\pi(\varphi')} \right] \pi(\varphi') g_c(\varphi'|\varphi) d\varphi', \quad (59)$$

which yields the following solution:

$$\alpha = \frac{1}{\delta + \theta}, \quad \beta = \frac{\theta}{\delta + \theta}, \quad \eta(\varphi') = \frac{\pi(\varphi') + \theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\psi) \pi(\psi) g_c(\psi|\varphi') d\psi}{(\delta + \theta) \pi(\varphi')}. \quad (60)$$

Define the following variables and functions:  $x \equiv \varphi'$ ,  $y \equiv \psi$ ,  $f(x) \equiv \eta(\varphi') \pi(\varphi')$ ,  $f(y) \equiv \eta(\psi) \pi(\psi)$ ,  $K(x, y) \equiv g_c(\psi|\varphi')$ ,  $a(x) \equiv \pi(\varphi') / (\delta + \theta)$ , and  $\mu \equiv \theta / (\delta + \theta)$ . Using these definitions, the expression for  $\eta(\varphi') \pi(\varphi')$  implied by equation (60) can be re-written as follows:

$$f(x) = (\Omega(f)(x)) \equiv a(x) + \mu \int_{\varphi^*}^{\bar{\varphi}} f(y) K(x, y) dy, \quad (61)$$

which takes the form of a *Fredholm* integral equation. The continuous functions  $K(x, y)$  and  $a(x)$  are known. For all  $x$  and  $y$  in the intervals  $\varphi^* \leq x \leq \bar{\varphi}$ ,  $\varphi^* \leq y \leq \bar{\varphi}$ , we have  $|K(x, y)| \leq \bar{K} < 1$ , as  $K(x, y)$  is a continuous probability density function. Under the assumption  $\mu \bar{K} |\bar{\varphi} - \varphi^*| < 1$ , the mapping  $\Omega$  is a contraction, since (see for example Marsden and Hoffman 1974):

$$d(\Omega(f_1), \Omega(f_2)) = \sup_x \left| \mu \int_{\varphi^*}^{\bar{\varphi}} K(x, y) [f_1(y) - f_2(y)] dy \right| \leq \mu \bar{K} |\bar{\varphi} - \varphi^*| d(f_1, f_2), \quad (62)$$

where  $\mu \bar{K} |\bar{\varphi} - \varphi^*| < 1$  is satisfied for  $\theta < \bar{\theta}$ , where  $\bar{\theta}$  is implicitly defined by  $\frac{\bar{\theta}}{\delta + \bar{\theta}} \equiv \frac{1}{\bar{K} |\bar{\varphi} - \varphi^*|}$ . In determining the stationary distribution of firm productivity below, we will assume that the parameter condition  $\theta < \bar{\theta}$  is satisfied. Therefore, the Fredholm equation (61) has a unique solution, which determines the equilibrium value of  $f(x) \equiv \eta(\varphi') \pi(\varphi')$ . Since  $\pi(\varphi')$  is a known function, we have determined the unique equilibrium value of  $\eta(\varphi')$ .

### F3. Proof of Proposition 1

**Proof.** The equality of inflows and outflows for each productivity in (19) can be re-written as follows:

$$\gamma_g(\varphi) = \frac{g_e(\varphi) M_e}{(\delta + \theta) M} + \left( \frac{\theta}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} g_c(\varphi|\varphi') \gamma_g(\varphi') d\varphi' \quad (63)$$

where  $\{M_e, M, \varphi^*\}$  are aggregate variables and we have used the fact that the second and third terms on the right-hand side of (19) sum to  $\theta \gamma_g(\varphi) M$ .

Define the following variables and functions:  $x \equiv \varphi$ ,  $y \equiv \varphi'$ ,  $f(x) \equiv \gamma_g(\varphi)$ ,  $f(y) \equiv \gamma_g(\varphi')$ ,  $K(x, y) \equiv g_c(\varphi|\varphi')$ ,  $a(x) \equiv g_e(\varphi) M_e / ((\delta + \theta) M)$ ,  $\mu \equiv (\theta / (\delta + \theta))$ . Using these definitions, equation (63) can be re-expressed in the following way:

$$f(x) = (\Omega(f)(x)) \equiv a(x) + \mu \int_{\varphi^*}^{\bar{\varphi}} K(x, y) f(y) dy, \quad (64)$$

which takes the form of a *Fredholm* integral equation. The continuous functions  $K(x, y)$  and  $a(x)$  are known. For all  $x$  and  $y$  in the intervals  $\varphi^* \leq x \leq \bar{\varphi}$ ,  $\varphi^* \leq y \leq \bar{\varphi}$ , we have

$|K(x, y)| \leq \bar{K} < 1$ , as  $K(x, y)$  is a continuous probability density function. Under the assumption  $\mu \bar{K} |\bar{\varphi} - \varphi^*| < 1$ , the mapping  $\Omega$  is a contraction, since (see for example Marsden and Hoffman 1974):

$$d(\Omega(f_1), \Omega(f_2)) = \sup_x \left| \mu \int_{\varphi^*}^{\bar{\varphi}} K(x, y) [f_1(y) - f_2(y)] dy \right| \leq \mu \bar{K} |\bar{\varphi} - \varphi^*| d(f_1, f_2), \quad (65)$$

where, from the above,  $\mu \bar{K} |\bar{\varphi} - \varphi^*| < 1$  is satisfied for  $\theta < \bar{\theta}$ , where  $\bar{\theta}$  is implicitly defined by  $\frac{\bar{\theta}}{\delta + \bar{\theta}} \equiv \frac{1}{\bar{K} |\bar{\varphi} - \varphi^*|}$ . For parameters satisfying this condition, the Fredholm equation (64) has a unique solution, which defines the unique stationary distribution of productivity conditional upon entry,  $f(x) \equiv \gamma_g(\varphi)$ , for each value of  $\{M_e, M, \varphi^*\}$ . ■

#### F4. Proof of Proposition 2

**Proof.** The equality of inflows and outflows for each value of consumer tastes in (11) can be re-written as follows:

$$\begin{aligned} \gamma_{zi}(\lambda_i) &= \frac{z_{ei}(\lambda_i) [1 - G_e(\varphi^*)] M_e}{(\delta + \chi + \varepsilon_i) M} \\ &+ \left( \frac{\varepsilon_i}{\delta + \chi + \varepsilon_i} \right) \left[ \int_{\underline{\lambda}}^{\bar{\lambda}} z_{ic}(\lambda_i | \lambda'_i) \gamma_{zi}(\lambda'_i) d\lambda'_i \right] \end{aligned} \quad (66)$$

where  $\{M_e, M, \varphi^*\}$  are again aggregate variables and the functions  $\{G_e(\varphi^*), z_{ei}(\lambda_i), z_{ic}(\lambda_i | \lambda'_i)\}$  are known. The function  $\chi \equiv \theta \int_{\varphi^*}^{\bar{\varphi}} G_c(\varphi^* | \varphi) \gamma_g(\varphi) d\varphi$  depends upon  $G_c(\varphi^* | \varphi)$ , which is known, and  $\gamma_g(\varphi)$ , which is determined in the proof of Proposition 1 above.

Define the following variables and functions:  $x \equiv \lambda_i$ ,  $y \equiv \lambda'_i$ ,  $f(x) \equiv \gamma_{zi}(\lambda_i)$ ,  $f(y) \equiv \gamma_{zi}(\lambda'_i)$ ,  $K(x, y) \equiv z_{ic}(\lambda_i | \lambda'_i)$ ,  $a(x) \equiv z_{ei}(\lambda_i) [1 - G_e(\varphi^*)] M_e / ((\delta + \chi + \varepsilon_i) M)$ ,  $\mu \equiv \varepsilon_i / (\delta + \chi + \varepsilon_i)$ . Using these definitions, equation (66) can be re-expressed as follows:

$$f(x) = (\Omega(f))(x) \equiv a(x) + \mu \int_{\underline{\lambda}}^{\bar{\lambda}} K(x, y) f(y) dy \quad (67)$$

which again takes the form of a *Fredholm* integral equation. The continuous functions  $K(x, y)$  and  $a(x)$  are known. For all  $x$  and  $y$  in the intervals  $\underline{\lambda} \leq x \leq \bar{\lambda}$ ,  $\underline{\lambda} \leq y \leq \bar{\lambda}$ , we have  $|K(x, y)| \leq \bar{K} < 1$ , as  $K(x, y)$  is a continuous probability density function. Under the assumption  $\mu \bar{K} |\underline{\lambda} - \bar{\lambda}| < 1$ , the mapping  $\Omega$  is a contraction (see for example Marsden and Hoffman 1974). The condition  $\mu \bar{K} |\underline{\lambda} - \bar{\lambda}| < 1$  is satisfied for  $\varepsilon_i < \bar{\varepsilon}$ , where  $\bar{\varepsilon}$  is implicitly defined by  $\frac{\bar{\varepsilon}}{\delta + \chi + \bar{\varepsilon}} \equiv \frac{1}{\bar{K} |\underline{\lambda} - \bar{\lambda}|}$ . For parameters satisfying this condition, the Fredholm equation (67) has a unique solution, which defines a unique stationary distribution of consumer tastes,  $f(x) \equiv \gamma_{zi}(\lambda_i)$ , for each value of  $\{M_e, M, \varphi^*\}$ . ■

#### F5. Proof of Proposition 3

**Proof.** We first determine the equilibrium values of  $\{P_i, R_i\}$  given  $\{\varphi^*, \lambda_i^*(\varphi^*)\}$  and given the stationary distributions  $\{\gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ . The product price indices  $P_i$  in equation (24) depend on weighted average productivity  $\tilde{\varphi}_i$  and the mass of firms producing the product  $M_{pi}$ . From equations (26), (56) and (57),  $\tilde{\varphi}_i$  is uniquely determined by  $\{\varphi^*, \lambda_i^*(\varphi^*), \gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ . Similarly, it can be shown that  $M_{pi}$  is uniquely determined by  $\{\varphi^*, \lambda_i^*(\varphi^*), \gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ . To do so, note that  $M_{pi}$  is a constant fraction of the mass of firms producing  $M$ . From equation (21),

this constant fraction depends only on  $\{\varphi^*, \lambda_i^*(\varphi^*), \gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ , and from (22)  $M = R/\bar{r}$ . From equations (10), (13), (26) and (22),  $\bar{r}$  is uniquely determined by  $\{\varphi^*, \lambda_i^*(\varphi^*), \gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ , and we have shown that  $R = L$ . Therefore the product price indices  $P_i$  are uniquely determined by  $\{\varphi^*, \lambda_i^*(\varphi^*), \gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ . With the product price indices determined, aggregate revenue for each product  $R_i$  can be solved for from equation (31) using  $R = L$ .

We next determine the stationary distributions  $\{\gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$ . From the proof of Propositions 1 and 2, for  $\theta < \bar{\theta}$  and  $\varepsilon_i < \bar{\varepsilon}$ , there exist unique stationary distributions for firm productivity,  $\gamma_g(\varphi)$ , and consumer tastes,  $\gamma_{zi}(\lambda_i)$ , for each value of the aggregate variables  $\{M_e, M, \varphi^*\}$ . The equilibrium value of  $M = R/\bar{r}$  was determined above, and the equilibrium value of  $M_e$  can be determined from  $M$  using equation (23).

Finally, we determine the equilibrium values of  $\{\varphi^*, \lambda_i^*(\varphi^*)\}$ . The equilibrium value of  $\lambda_i^*(\varphi^*)$  can be determined given  $\varphi^*$  from equation (27), where  $P_i$  and  $R_i$  were determined above. The equilibrium value of  $\varphi^*$  can be determined given  $\lambda_i^*(\varphi^*)$  and the stationary distributions  $\{\gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$  from the free entry condition (28). Note that  $\lambda_i^*(\varphi) = (\varphi^*/\varphi) \lambda_i^*(\varphi^*)$ . Therefore, from equation (13),  $\pi(\varphi) \rightarrow \infty$  as  $\varphi^* \rightarrow 0$  and  $\pi(\varphi) \rightarrow 0$  as  $\varphi^* \rightarrow \infty$ . Hence, in the free entry condition (28),  $V \rightarrow \infty$  as  $\varphi^* \rightarrow 0$ , and  $V \rightarrow 0$  as  $\varphi^* \rightarrow \infty$ . Since  $V$  is continuous in  $\varphi^*$ , it follows that there exists an equilibrium value of  $\varphi^*$  for which the expected value of entry  $V$  equals the sunk entry cost  $f_e$ . Differentiating  $V$  with respect to  $\varphi^*$  given  $\lambda_i^*(\varphi^*)$  and  $\{\gamma_g(\varphi), \gamma_{zi}(\lambda_i)\}$  yields:

$$\begin{aligned} \frac{\partial V}{\partial \varphi^*} = & \underbrace{- \left( \frac{1}{\delta + \theta} \right) \left[ \pi(\varphi^*) + \theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') g_c(\varphi'|\varphi^*) d\varphi' \right]}_{\text{Term A}} g_e(\varphi^*) \\ & + \left( \frac{1}{\delta + \theta} \right) \int_{\varphi^*}^{\bar{\varphi}} \left[ \underbrace{\frac{\partial \pi(\varphi)}{\partial \varphi^*}}_{\text{Term B}} + \underbrace{\theta \int_{\varphi^*}^{\bar{\varphi}} \frac{\partial \eta(\varphi')}{\partial \varphi^*} \pi(\varphi') g_c(\varphi'|\varphi) d\varphi'}_{\text{Term C}} \right. \\ & \quad \left. + \underbrace{\theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \frac{\partial \pi(\varphi')}{\partial \varphi^*} g_c(\varphi'|\varphi) d\varphi'}_{\text{Term D}} \right. \\ & \quad \left. + \underbrace{\theta \int_{\varphi^*}^{\bar{\varphi}} \eta(\varphi') \pi(\varphi') \frac{\partial g_c(\varphi'|\varphi)}{\partial \varphi^*} d\varphi'}_{\text{Term D}} \right. \\ & \quad \left. + \underbrace{-\theta \eta(\varphi^*) \pi(\varphi^*) g_c(\varphi^*|\varphi)}_{\text{Term E}} \right] g_e(\varphi) d\varphi \\ & \quad \underbrace{\hspace{10em}}_{\text{Term F}} \end{aligned}$$

where Term A = 0 since  $v(\varphi^*) = 0$ ; Term B < 0 since  $\partial \pi(\varphi) / \partial \varphi^* < 0$  from equations (13) and (13); and Term F < 0. Term D < 0 since  $\partial \pi(\varphi) / \partial \varphi^* < 0$  from equations (13) and (13). In Term E,  $\partial g(\varphi'|\varphi) / \partial \varphi^* = 0$  for  $\varphi \neq \varphi^*$ , and so Term E is of measure zero relative to Term D. Finally, as  $\theta \rightarrow 0$ ,  $\eta(\varphi') \rightarrow 1/\delta$ ,  $\partial \eta(\varphi') / \partial \varphi^* \rightarrow 0$ , and Term C  $\rightarrow 0$ . Therefore, for sufficiently small  $\theta < \bar{\theta}$ ,  $V$  is monotonically decreasing in  $\varphi^*$ , and there exists a unique equilibrium value of  $\varphi^*$ . ■

## References

Bernard, Andrew B., Stephen J. Redding and Peter K. Schott (2007) ‘‘Comparative Advantage and Heterogeneous Firms,’’ *Review of Economic Studies*, 74, 31-66.

- Ericson, R. and A. Pakes (1995) "Markov-Perfect Industry Dynamics: A Framework for Empirical Work", *Review of Economic Studies*, 62, 53-82.
- Foster, L., J. Haltiwanger and C. Syverson (2008) "Reallocation, Firm Turnover and Efficiency: Selection on Productivity or Profitability," *American Economic Review*, 98(1), 394-425.
- Hopenhayn, H. (1992) "Entry, Exit, and Firm Dynamics in Long Run Equilibrium", *Econometrica*, 60(5), 1127-1150.
- Jovanovic, B. (1982) "Selection and the Evolution of Industry", *Econometrica*, 50(3), 649-70.
- Klette, T. J. and Z. Griliches (1996) "The Inconsistency of Common Scale Estimators when Output Prices are Unobserved and Endogenous," *Journal of Applied Econometrics*, 11, 343-361.
- De Loecker, J. (2008) "Product Differentiation, Multi-Product Firms and Estimating the Impact of Trade Liberalization on Productivity," Princeton University, mimeograph.
- Levinsohn, J. and M. J. Melitz (2006) "Productivity in a Differentiated Products Market Equilibrium," Princeton University, mimeograph.
- Marsden, J. and M. Hoffman (1974) *Elementary Classical Analysis*, New York: Freeman, Second Edition.
- Melitz, M. J. (2003) "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity", *Econometrica*, 71, 1695-1725.
- Melitz, M. J. and G. I. Ottaviano (2008) "Market Size, Trade, and Productivity," *Review of Economic Studies*, 75(1), 295-316.