Arbitrage Pricing Theory (APT)

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Basic assumptions

- The CAPM assumes homogeneous expectations and mean-variance preferences.
  - The result: The model identifies the market portfolio as the only risk factor.
- The APT makes no assumption about expectations or investor risk preferences.
  - Consequently, the model does not identify any risk factor.
- The CAPM and the APT both require perfectly competitive securities markets.
“No arbitrage opportunities”

- In the markets underlying both the CAPM and the APT, there are no opportunities for making arbitrage profits
  - This means that two securities with identical payoffs in all states must have the same price today ("Law of one price")
  - It also means that riskless investment opportunities earn the riskless rate of return.
  - Zero-investment, riskless cash flows are eliminated through arbitrage activity

“Linear pricing”

- It can be shown mathematically that the absence of arbitrage opportunities in the market implies that the expected return on any asset is a linear function of the expected return on priced risk factors

- Because the theory does not identify what the prices factors are, we posit an arbitrary set of $K$ factors.

- The realized value of the $k$’th factor is $F_k$
(1) \[ r_i = E(r_i) + \beta_{i1}f_1 + \beta_{i2}f_2 + \ldots + \beta_{iK}f_K + e_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik}f_k + e_i \]

- Equation (1) is a K-factor return generating process
- \( E(r_i) \) = expected return on investment \( i \)
- \( f_k \) = k’th factor shock: \( F_k - E(F_k) \), where \( F_k \) is the realized factor value and \( E(f_k) = 0 \)
- \( \beta_{ik} \) = security \( i \)’s sensitivity with respect to the k’th factor (factor loading, or factor risk)
- \( e_i \) = firm-specific risk

\[ E(\beta) - r_F = \sum_{k=1}^{K} \beta_{ik} \lambda_k \]

- Equation (2) is the APT model
- \( r_F \) = the return on the risk-free asset
- \( \lambda_k \) = the risk premium of factor \( k \) (expected return on the k’th factor in excess of the risk-free return)
Equation (3) combines (1) and (2)

It says that the realized excess return on investment $i$ is generated by $i$’s exposure to factor risks and the unexpected factor realization, plus firm-specific risk.

**Problem:**
- Even if you know what the relevant factors $F_k$ are, you probably do not know $E(F_k)$ and it is therefore also difficult to estimate $f_k$

**Solution:** “factor mimicking”
- Form a broad portfolio that has a $\beta_k=1$ on factor $k$ while being independent of ($\beta_k=0$) the other $K-1$ factors
- This is called a “factor mimicking portfolio”
- Since this portfolio must also obey APT in (2), its expected excess return, $E(r_k)$, equals $\lambda_k$
- Plug this back into equation (3) and you have a return generating process stated purely in terms of observables.
Mimicking portfolio for factor k
- Portfolio k mimics a factor if
  - \( \beta_k = 1, \beta_j \neq k = 0 \)
- Price portfolio k using APT in (2):
  - \( E(r_k) = r_F + \lambda_k \)
- Shocks to factor portfolio k:
  - \( f_k = r_k - E(r_k) = r_k - (\lambda_k + r_F) \)
- Plug this shock into the right-hand side of the regression equation (3):
  - \( \beta_{ik}(\lambda_k + f_k) = \beta_{ik}(\lambda_k + r_k - \lambda_k - r_F) = \beta_{ik}(r_k - r_F) \)
- Do the same for all K factors, and you get..

\[
(3') \quad r_i - r_F = \sum_{k=1}^{K} \beta_{ik} (r_k - r_F) + e_i
\]

Equation (3’) says that the realized excess return on asset i is generated by i’s sensitivity to the realized excess returns (i.e., the realized risk premiums) on a set of K factor-mimicking portfolios plus the firm-specific return realization.
Equation (4) shows that the CAPM is a special case of the APT model (2) with only one factor, the return on value-weighted market portfolio $M$.

\begin{align*}
(4) \quad E(r_i) - r_F &= \beta_{iM} \lambda_M = \beta_{iM} \left[ E(r_M) - r_F \right]
\end{align*}

- Equation (5) is obtained by rewriting the RHS of (4).
- This is a portfolio with weights $(1-\beta_{iM})$ in the risk-free asset and $\beta_{iM}$ in the market $M$.
- We call this portfolio a tracking portfolio for investment $i$ since $E(r_i) = E(r_{Track})$.
- This tracking portfolio is exposed to factor risk only.

\begin{align*}
(5) \quad (1 - \beta_{iM})r_F + \beta_{iM} E(r_M) &\equiv E(r_{Track})
\end{align*}
What is the intuition behind this equation?

- Suppose the equality does not hold. Then, sell short one and invest the proceeds in the other, generating a positive risk-free cash inflow with no net investment, i.e., an arbitrage profit.
- Competition to capture this arbitrage profit restores the equality above

Note: Investment \( i \) has a positive net present value only if \( E(r_i) > E(r_{Track}) \)

\[ E(r_i) = E(r_{Track}) \]

(6) generalizes (5) to \( K \) factors, where \( \lambda_k + r_F \) replaces \( E(r_k) \)

- The idea of a tracking portfolio is central to understanding how the market values an investment. In principle, any investment can be tracked
- Thus, project valuation comes down to factor sensitivities of cash flows and of factor prices in an arbitrage-free environment
\[
\begin{align*}
  r_i &= E(r_i) + \beta_{i1}f_1 + \beta_{i2}f_2 + e_i \\
  r_A &= .03 + 1f_1 - 4f_2 + e_A \\
  r_B &= .05 + 3f_1 + 2f_2 + e_B \\
  r_C &= .10 + 1.5f_1 + 0f_2 + e_C 
\end{align*}
\]

**Example 1:** Track the Wilshire 5000 stock index with stocks (A,B,C)
- Suppose Wilshire has $\beta_{W1}=2$, $\beta_{W2}=1$

\[
\begin{align*}
  \text{Factor 1} & : 1x_A + 3x_B + 1.5x_C = 2 \\
  \text{Factor 2} & : -4x_A + 2x_B + 0x_C = 1 \\
  \text{Portfolio} & : x_A + x_B + x_C = 1 
\end{align*}
\]

**Find a set of portfolio weight for the tracking portfolio that satisfies the above system of three equations**
- The solution is $x_A = -.1$, $x_B = .3$, $x_C = .8$
Tracking with Pure Factor Portf.

- While the portfolio in example 1 tracks the Wilshire 5000, the tracking portfolio is likely to carry substantial firm-specific risk which the Wilshire 5000 does not.
- The solution is to use well diversified portfolios as the underlying assets in the tracking portfolio.

Pure factor Portfolios (PFPs)

- A PFP is a well diversified portfolio that tracks a given factor and that is also independent of all other factors.
  - Thus, a PFP for factor $k$ has $\beta_{PFP_k} = 1$ and $\beta_{PFP} = 0$ for all the remaining $K-1$ factors.
  - With $K$ factors, one can use any set of $K+1$ investments that lack firm-specific risk (are well diversified) to create $K$ PFPs.
\[ r_i = E(r_i) + \beta_{i1}f_1 + \beta_{i2}f_2 + e_i \]
\[ r_c = .08 + 2f_1 + 3f_2 + e_c \]
\[ r_g = .10 + 3f_1 + 2f_2 + e_g \]
\[ r_s = .10 + 3f_1 + 5f_2 + e_s \]

- **Example 2**: Find the PFPs for \( f_1 \) and \( f_2 \) using the three stocks \( c, g, s \)
  - Need to solve two systems of linear equations, one for each PFP

**PFP 1**: 
\[ 2x_c + 3x_g + 3x_s = 1 \]
\[ 3x_c + 2x_g + 5x_s = 0 \]
\[ x_c + x_g + x_s = 1 \]
\[ \Rightarrow x_c = 2, \quad x_g = 1/3, \quad x_s = -4/3 \]

**PFP 2**: 
\[ 2x_c + 3x_g + 3x_s = 0 \]
\[ 3x_c + 2x_g + 5x_s = 1 \]
\[ x_c + x_g + x_s = 1 \]
\[ \Rightarrow x_c = 3, \quad x_g = -2/3, \quad x_s = -4/3 \]

- What are the factor equations for the PFPs?
\[ r_{PPk} = E(r_{PPk}) + f_k \quad k = 1,2 \]

\[ E(r_{PPk}) = \sum_{n=1}^{N} x_{PPnk} E(r_n) \]

\[ E(r_{PP1}) = (2)(.08) + (1/3)(.10) - (4/3)(.10) = .06 \]
\[ E(r_{PP2}) = (3)(.08) - (2/3)(.10) - (4/3)(.10) = .04 \]

**Thus:**
\[ r_{PP1} = .06 + f_1 \]
\[ r_{PP2} = .04 + f_2 \]

Also, compute factor premiums from equation (6)
\[ E(r_{PPk}) = (1 - \sum_{k=1}^{K} \beta_{PPk})r_F + \sum_{k=1}^{K} \beta_{PPk}(\lambda_k + r_F) = \lambda_k + r_F \]

If \( r_F = .05 \), then \( \lambda_1 = .01 \), \( \lambda_2 = -.01 \)

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- So, you track an investment with \( \beta_1 = .25 \), \( \beta_2 = .5 \) simply by holding a portfolio consisting of PFP1, PFP2 and the risk-free asset, with portfolio weights
  - \( x_{PFP1} = .25 \)
  - \( x_{PFP2} = .5 \)
  - \( x_F = 1 - (.25 + .5) = .25 \)

- In general, with K factors, an investment with no firm-specific risk and a factor beta of \( \beta_k \) on the k'th factor is tracked by a portfolio with weights
  - \( \beta_k \) on the pure factor portfolio for \( k \) and
  - \( (1 - \sum_k \beta_k) \) on the risk-free asset
• With no firm-specific risk, the factor equations of the tracking portfolio and the tracked investment will be identical with the possible exception of the intercept terms.

• Differences in the intercept terms represent differences in expected returns.

• If the expected returns differ, short the one with the smallest intercept, and go long in the other.

• This generates arbitrage profits (positive net present value) for the investment.

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**Factor Models and Asset Allocation**

• Suppose there are 8,000 traded stocks.

  - The full variance-covariance matrix of these stocks contains 64 million elements.

  - The use of factor models greatly simplifies the estimation of these variances and covariances.
The simplification occurs because the factors are uncorrelated with each other and with firm-specific risks (the variance-covariance matrix is diagonal). Moreover, firm-specific risks are uncorrelated across individual securities. We have that:

\[ r_i = E(r_i) + \beta_{i1} f_1 + \beta_{i2} f_2 + \cdots + \beta_{ik} f_k + e_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} f_k + e_i \]

(7) \[ \text{Var}(r_i) = \sum_{k=1}^{K} \beta_{ik}^2 \text{Var}(F_k) + \text{Var}(e_i) \]

(8) \[ \text{Cov}(r_i, r_j) = \sum_{k=1}^{K} \beta_{ik} \beta_{jk} \text{Var}(F_k) \]

- Thus, with 8,000 stocks and, e.g., five factors, you need only to estimate 40,000 stock betas, five factor variances and 8,000 firm-specific variances to reproduce the full variance-covariance matrix.
- This explains the popularity of factor models in asset allocation and investment decisions in practice.
- For well diversified portfolios, the firm-specific variance component is close to zero, so that the total variance is driven purely by factor risk.
Identifying the Factors

- The APT does not identify the factors
- One method to identify factors is to apply principal-components analysis to the variance-covariance matrix of security returns. The resulting factors are, however, difficult to interpret and may have “strange” portfolio weights
- Our approach: Use economic theory to prespecify a set of “reasonable” macro-economic factors and see what works

Frequently used macro-economic factors:
- The market index
- Unexpected growth in industrial production (or unexpected changes in the business cycle)
- Changes in expected and unexpected inflation (changes in expected inflation proxied by the change in the T-bill rate)
- Unexpected changes in default spread: unexpected changes in the spread between AAA-rated and BAA-rated corporate bond returns
- Unexpected changes in the term spread as measured by the difference between long and short government bonds